A Landau fluid model for warm collisionless plasmas

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A Landau fluid model for a collisionless electron-proton magnetized plasma, that accurately reproduces the dispersion relation and the Landau damping rate of all the magnetohydrodynamic waves, is presented. It is obtained by an accurate closure of the hydrodynamic hierarchy at the level of the fourth order moments, based on linear kinetic theory. It retains non-gyrotropic corrections to the pressure and heat flux tensors up to the second order in the ratio between the considered frequencies and the ion cyclotron frequency.

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I. INTRODUCTION

In many spatial and astrophysical plasmas, collisions are negligible, making the usual magnetohydrodynamics (MHD) questionable. The presence of a strong ambient magnetic field nevertheless ensures a collective behavior of the plasma, making a hydrodynamic approach of the large-scale dynamics possible and even advantageous, compared with purely kinetic descriptions provided by the Vlasov-Maxwell (VM) or the gyrokinetic equations. It is thus of great interest, both for the numerical simulation of broad spectrum phenomena and for an easier interpretation of the involved processes, to construct fluid models that extend the MHD equations to collisionless situations by including finite Larmor radius (FLR) corrections and Landau damping. In a fluid formalism, FLR corrections refer to the part of the pressure and heat flux tensors associated with the deviation from gyrotropy. They play a role when the transverse scales under consideration extend up to or beyond the ion Larmor radius (fluid models are always limited to parallel scales large compared with the ion Larmor radius). Evolving on a shorter time scale than the basic hydrodynamic fields, FLR corrections can generally be computed perturbatively. This expansion cannot however be pushed arbitrary far and any fluid analysis addressing (transverse) scales comparable to the ion Larmor radius[1] can only be heuristic.

From Vlasov equation it is easy to derive a set of exact moment equations. This fluid hierarchy is however faced with a closure problem. An interesting approach consists in closing this hierarchy by using relations, derived from linearized kinetic theory, between given moments and lower order ones. This in particular accounts for linear Landau damping in a fluid formalism. Such an approach initiated in Ref. [2] leads to descriptions usually referred to as Landau fluids. We here concentrate on a closure at the level of the fourth order moments, which provides an accurate description of most of the usual hydrodynamic quantities.

An alternative method to the Landau fluids is provided by the gyrofluids[3,4] obtained by taking the moments of gyrokinetic equations. The same closure problem is encountered for the moment hierarchy. The gyrofluids have the advantage of retaining FLR corrections to all order relatively to the transverse scale within a low frequency asymptotics but, being written in a local reference frame, the resulting equations are more complex than those governing the Landau fluids, we are here concerned with.

As an example, Landau fluid models should be most useful to analyze the dynamics of the magnetosheath that appears as a buffer between the earth bow shock and the magnetopause and plays an important role in decreasing the impact of solar activity on the earth environment. Recent analyses of data provided by the Cluster spacecraft mission have revealed that the magnetosheath displays a wide spectrum of low frequency modes (Alfvén, slow and fast magnetosonic, mirror)[5] whose wavelengths extend down to the ion gyroradius and beyond. Since the plasma is relatively warm and collisionless, Landau damping and FLR corrections are supposed to play an important role. Coherent solitonic structures (magnetic holes and shocklets) are also observed, and their origin is still debated.[6,7]

A Landau fluid model for collisionless purely magnetohydrodynamic regimes[8] was first derived from the equation for the distribution function of the particle guiding centers, taken to lowest order. It is thus restricted to the largest MHD scales where the pressure and heat flux tensors for each species can be viewed as gyrotropic and where the transverse velocity reduces to the electric drift. Starting directly from the VM equations, this model was then extended in order to include a generalized Ohm's law and to retain the leading order FLR corrections to the pressure tensor.[9,10] This model enabled one to reproduce the dynamics of dispersive Alfvén waves propagating along the ambient field both in the linear and weakly-nonlinear regimes and to recover the kinetic derivative nonlinear Schrödinger (KDNLS) equation in a long-wave asymptotic expansion with, as the only difference, the replacement of the plasma response function by its two or four poles Padé approximants. It also accurately describes the dissipation of oblique magnetosonic waves.[11] Non-gyrotropic contributions to the heat fluxes were introduced in Ref. [12] in order to obtain the dispersion relation and the Landau damping rate of oblique and kinetic Alfvén waves. The approach we present here provides a more systematic description of the FLR corrections up to second order, by retaining parallel and transverse heat flux vectors whose coupling to the non-gyrotropic pressure contributions is in particular required for an accurate description of the transverse magnetosonic waves.[13] A recent paper by Ramos[14] addresses a similar issue and derives a complete set of nonlinear equations for fluid moments up to the heat flux vectors, leaving the closure on the fourth order moments unspecified. We here follow a similar path choosing in Section II to linearize the equations for the ("slaved") non-gyrotropic contributions to the pressure and heat flux tensors, while retaining nonlinear equations for all the other moments. While Ramos performs a first order expansion in the regime referred to as the fast dynamics ordering, we here keep the second order accuracy necessary for a proper description of the oblique dynamics. By fitting with the kinetic theory briefly reported in Section III, we also give in Section IV an explicit closure relation, taking into account FLR corrections, and approximating the plasma response function with four and three poles Padé approximants in order to recover accurate limits for Landau damping both in the isothermal and adiabatic regimes. As the result of such a high order approximation, one of the fourth order moments is prescribed as the solution of a dynamical equation. After a discussion of the resulting model in Section V, the validation of the model at the level of the dispersion relation of the various MHD waves is addressed in Section VI. Section VII is the conclusion where further extensions to a model, aimed at including a realistic description of the mirror modes, are announced.

II. FLUID DESCRIPTION OF EACH PARTICLE SPECIES

A. The moment hierarchy

Starting from the VM equations for the distribution function f_r of the particles of species r with charge q_r, mass m_r, and average number density n_r, one easily derives a hierarchy of fluid equations for the corresponding density $\rho_r = m_r n_r \int f_r d^3v$, hydrodynamic velocity $u_r = \int v f_r d^3v / \int f_r d^3v$, pressure tensor $\mathbf{p}_r = m_r n_r \int (v-u_r)\otimes(v-u_r) f_r d^3v$ and heat flux tensor $\mathbf{q}_r = m_r n_r \int (v-u_r)\otimes(v-u_r)\otimes(v-u_r) f_r d^3v$, in the usual form

$$\partial_t \rho_r + \nabla\cdot(\rho_r u_r) = 0 \tag{1}$$

$$\partial_t u_r + u_r\cdot\nabla u_r + \frac{1}{\rho_r}\nabla\cdot\mathbf{p}_r - \frac{q_r}{m_r}(E + \frac{1}{c}u_r\times B) = 0 \tag{2}$$

$$\partial_t \mathbf{p}_r + \nabla\cdot(u_r\mathbf{p}_r + \mathbf{q}_r) + \Big[\mathbf{p}_r\cdot\nabla u_r + \frac{q_r}{m_r c} B\times\mathbf{p}_r\Big]^{\mathcal{S}} = 0, \tag{3}$$

where the tensor $B\times\mathbf{p}_r$ has elements $(B\times\mathbf{p}_r)_{ij} = \epsilon_{iml} B_m p_{r\,lj}$ and where, for a square matrix $\mathbf{a}$, one defines $\mathbf{a}^{\mathcal{S}} = \mathbf{a} + \mathbf{a}^{\mathrm{tr}}$. One has $(B\times\mathbf{p}_r)^{\mathrm{tr}} = -\mathbf{p}_r\times B$. In order to distinguish between scalar and tensorial pressures, bold letters are used to denote tensors of rank two and higher. The equation for the heat flux tensor involves the fourth order moment $\mathbf{r}_r = m_r n_r \int (v-u_r)\otimes(v-u_r)\otimes(v-u_r)\otimes(v-u_r) f_r d^3v$. Since at this step we are dealing with the various particle species separately, we simplify the writing by hereafter dropping the r subscript. The equations governing the heat flux elements then read

$$\begin{aligned}&\partial_t q_{ijk} + v_l\partial_l q_{ijk} + \partial_l r_{ijkl} - \frac{1}{\rho}\partial_l p_{lm}(\delta_{mi}p_{jk} + \delta_{mj}p_{ik} + \delta_{mk}p_{ij})\\ &+\partial_l u_m(\delta_{mi}q_{jkl} + \delta_{mj}q_{ikl} + \delta_{mk}q_{ijl} + \delta_{ml}q_{ijk}) - \Omega\widehat{b}_n(\epsilon_{imn}q_{jkm} + \epsilon_{jmn}q_{ikm} + \epsilon_{kmn}q_{ijm}) = 0.\end{aligned} \tag{4}$$

We here concentrate on the ion dynamics. The corresponding equations for the electrons are obtained from the equations for the ions by changing the sign of the electric charge (including in the cyclotron frequency) and making the approximation $m_e/m_p \ll 1$.

B. Pressure tensors and heat flux vectors

In order to isolate the gyrotropic components of the pressure tensor, it is convenient to rewrite Eq. (3) for the pressure tensor of each particle species in the form

$$\mathbf{p}\times\widehat{b} - \widehat{b}\times\mathbf{p} = \mathbf{k} \tag{5}$$

where $\widehat{b} = \dfrac{B}{|B|}$ is the unit vector along the local magnetic field and

$$\mathbf{k} = \frac{1}{\Omega}\frac{B_0}{|B|}\Big[\frac{d\mathbf{p}}{dt} + (\nabla\cdot u)\mathbf{p} + \nabla\cdot\mathbf{q} + (\mathbf{p}\cdot\nabla u)^S\Big]. \tag{6}$$

In this equation, B_0 denotes the amplitude of the ambient field assumed to be oriented in the z-direction, and $\Omega = \dfrac{qB_0}{mc}$ is the cyclotron frequency of the considered particles species with charge q and mass m. Furthermore, $\dfrac{d}{dt} = \partial_t + u\cdot\nabla$ denotes the convective derivative.

We first note that the left-hand side of Eq. (5) can be viewed as a self-adjoint linear operator acting on $\mathbf{p}$, whose kernel is spanned by the tensors $\mathbf{n} = \mathbf{I} - \widehat{b}\otimes\widehat{b}$ and $\boldsymbol{\tau} = \widehat{b}\otimes\widehat{b}$. Using the symbol : to denote double contraction, it is convenient to define the projection $\overline{\mathbf{a}}$ of any (3×3) rank two tensor $\mathbf{a}$ on the image of this operator as $\overline{\mathbf{a}} = \mathbf{a} - \frac{1}{2}(\mathbf{a}:\mathbf{n})\mathbf{n} - (\mathbf{a}:\boldsymbol{\tau})\boldsymbol{\tau}$, which implies $\operatorname{tr}\overline{\mathbf{a}} = 0$ and $\overline{\mathbf{a}}:\boldsymbol{\tau} = 0$. In particular, the pressure tensor $\mathbf{p} = \mathbf{P} + \boldsymbol{\Pi}$ is written as the sum of a gyrotropic pressure $\mathbf{P} = p_\perp\mathbf{n} + p_\|\boldsymbol{\tau}$ (with $2p_\perp = \mathbf{p}:\mathbf{n}$ and $p_\| = \mathbf{p}:\boldsymbol{\tau}$) and of a gyroviscous stress $\boldsymbol{\Pi} = \overline{\mathbf{p}}$ that satisfies $\boldsymbol{\Pi}:\mathbf{n} = 0$ and $\boldsymbol{\Pi}:\boldsymbol{\tau} = 0$.

A similar decomposition is performed on the heat flux tensor by writing $\mathbf{q} = \mathbf{S} + \boldsymbol{\sigma}$ with the conditions $\sigma_{ijk}n_{jk} = 0$ and $\sigma_{ijk}\tau_{jk} = 0$. One has

$$\begin{aligned} S_{ijk} = {} & \frac{1}{2}\Big(S^\perp_i n_{jk} + S^\perp_j n_{ik} + S^\perp_k n_{ij} + S^\perp_l \tau_{li} n_{jk} + S^\perp_l \tau_{lj} n_{ik} + S^\perp_l \tau_{lk} n_{ij}\Big) \\ & + S^\|_i \tau_{jk} + S^\|_j \tau_{ik} + S^\|_k \tau_{ij} - \frac{2}{3}\Big(S^\|_l \tau_{li}\tau_{jk} + S^\|_l \tau_{lj}\tau_{ik} + S^\|_l \tau_{lk}\tau_{ij}\Big), \end{aligned} \tag{7}$$

where the parallel and transverse heat flux vectors $S^\|$ and $S^\perp$ have components $S^\|_i = q_{ijk}\tau_{jk}$ and $2S^\perp_i = q_{ijk}n_{jk}$. In the special case where the tensor $\mathbf{q}$ is gyrotropic, only the z-components $q_\| = S_\|\cdot\widehat{b}$ and $q_\perp = S_\perp\cdot\widehat{b}$ are non zero. Transverse components are however required, for example to describe transverse magnetosonic waves.[13]

We consider in this paper perturbations that are at large scale in all space directions and in time, with an amplitude that is relatively small. This leads us to retain the terms involving the non-gyrotropic parts of the pressure and heat flux tensors at the linear level only. Such an ordering implies in particular that increasing the amplitude of the fluctuations requires longer length scales for preserving a given accuracy. In the following, we shall thus neglect the σ contribution to the heat flux tensor. One indeed easily checks from the equation satisfied by $\boldsymbol{\sigma}$ (see Appendix 2 of Ref. [15]) that $\boldsymbol{\sigma}$ involves either nonlinear contributions or linear contributions of second order relatively to the scale separation parameter, and thus turns out to be negligible in the equations for the gyroviscous stress or for the heat fluxes, at the order of the present analysis.

C. Dynamics of the gyrotropic pressures

To obtain the equations for the gyrotropic pressure components, one applies the contraction with the tensors $\mathbf{I}$ and $\boldsymbol{\tau}$ on both sides of Eq. (5) to get[12,15]

$$\begin{aligned} & \partial_t p_\perp + \nabla\cdot(u\,p_\perp) + p_\perp\nabla\cdot u - p_\perp\,\widehat{b}\cdot\nabla u\cdot\widehat{b} + \frac{1}{2}\Big(\operatorname{tr}\nabla\cdot\mathbf{q} - \widehat{b}\cdot(\nabla\cdot\mathbf{q})\cdot\widehat{b}\Big) \\ & \quad + \frac{1}{2}\Big(\operatorname{tr}(\boldsymbol{\Pi}\cdot\nabla u)^S - (\boldsymbol{\Pi}\cdot\nabla u)^S:\boldsymbol{\tau} + \boldsymbol{\Pi}:\frac{d\boldsymbol{\tau}}{dt}\Big) = 0 \end{aligned} \tag{8}$$

$$\partial_t p_\| + \nabla\cdot(u\,p_\|) + 2p_\|\,\widehat{b}\cdot\nabla u\,\cdot\widehat{b} + \widehat{b}\cdot(\nabla\cdot\mathbf{q})\cdot\widehat{b} + (\boldsymbol{\Pi}\cdot\nabla u)^S:\boldsymbol{\tau} - \boldsymbol{\Pi}:\frac{d\boldsymbol{\tau}}{dt} = 0, \tag{9}$$

which appear as the condition for the solvability of Eq. (5). Note that it is important to retain the coupling to the gyroviscous stress (in spite of its smallness) in order to ensure energy conservation whatever the form of the forthcoming closure relations.[14]

Since $\boldsymbol{\sigma}$ does not contribute at a linear level in the pressure equations, we can neglect it and write

$$\widehat{b}\cdot(\nabla\cdot\mathbf{q})\cdot\widehat{b} \approx -2(\widehat{b}\cdot S^\perp)\nabla\cdot\widehat{b} + \nabla\cdot S^\| - 2\widehat{b}\cdot\nabla\widehat{b}\cdot S^\| \tag{10}$$

$$\frac{1}{2}\Big(\operatorname{tr}(\nabla\cdot\mathbf{q}) - \widehat{b}\cdot(\nabla\cdot\mathbf{q})\cdot\widehat{b}\Big) \approx \nabla\cdot S^\perp + (\widehat{b}\cdot S^\perp)\nabla\cdot\widehat{b} + \widehat{b}\cdot\nabla\widehat{b}\cdot S^\|. \tag{11}$$

D. Gyroviscous stress tensor

In order to determine the non-gyrotropic contributions to the pressure tensor of the various particle species, we start from Eq. (5) for the full pressure tensor. Using Eqs. (8)-(9) governing the gyrotropic pressures, Eq. (5) is rewritten

$$\mathbf{\Pi}\times\widehat{b}-\widehat{b}\times\mathbf{\Pi}=\overline{\boldsymbol{\kappa}}+\overline{L(\mathbf{\Pi})} \tag{12}$$

where

$$\boldsymbol{\kappa}=\frac{1}{\Omega}\frac{B_0}{|B|}\Big[\frac{d\mathbf{P}}{dt}+(\nabla\cdot u)\mathbf{P}+\nabla\cdot\mathbf{q}+(\mathbf{P}\cdot\nabla u)^{\mathcal{S}}\Big] \tag{13}$$

and

$$L(\mathbf{\Pi})=\frac{1}{\Omega}\frac{B_0}{|B|}\Big[\frac{d\mathbf{\Pi}}{dt}+(\nabla\cdot u)\mathbf{\Pi}+(\mathbf{\Pi}\cdot\nabla u)^{\mathcal{S}}\Big]. \tag{14}$$

The elements of $\boldsymbol{\kappa}$ rewrite

$$\begin{aligned}\overline{\kappa}_{ij}=&\frac{1}{\Omega}\frac{B_0}{|b|}\Big[(p_\parallel-p_\perp)\frac{d\tau_{ij}}{dt}+\overline{\partial_k q_{kij}}+p_\perp(n_{ik}\partial_k u_j+n_{jk}\partial_k u_i-n_{ij}n_{kl}\partial_l u_k)\\&+p_\parallel(\tau_{ik}\partial_k u_j+\tau_{jk}\partial_k u_i-2\tau_{ij}\tau_{kl}\partial_l u_k)\Big].\end{aligned} \tag{15}$$

Furthermore in Eq. (12), the element of the left-hand side with ij indices reads $\epsilon_{jkl}\Pi_{ik}b_l-\epsilon_{ikl}b_k\Pi_{lj}=b_l(\epsilon_{jkl}\Pi_{ik}+\epsilon_{ikl}\Pi_{kj})$, thus suggesting a misprint in Eq. (3.5) of Ref. [15]. When neglecting as previously the contribution originating from $\boldsymbol{\sigma}$, the heat flux term $\overline{\partial_k q_{kij}}$ reduces to

$$\overline{\partial_k q_{kij}}\approx\Big(\overline{\nabla\cdot S}\Big)_{ij}=\partial_k S_{kij}-\frac{1}{2}n_{ij}n_{mn}\partial_k S_{kmn}-\tau_{ij}\tau_{mn}\partial_k S_{kmn}. \tag{16}$$

In the linear approximation, we have

$$\begin{aligned}\overline{(\nabla\cdot\mathbf{S})}_{ij}=&\frac{1}{2}\Big[\partial_k\Big(S_i^\perp+(S^\perp\cdot\widehat{b})\,\widehat{b}_i\Big)n_{jk}+\partial_k\Big(S_j^\perp+(S^\perp\cdot\widehat{b})\,\widehat{b}_j\Big)n_{ik}\Big]\\&+\widehat{b}_j(\widehat{b}\cdot\nabla)S_i^\parallel+\widehat{b}_i(\widehat{b}\cdot\nabla)S_j^\parallel-2(\widehat{b}\cdot\nabla S^\parallel\cdot\widehat{b})\tau_{ij}-\frac{1}{2}(\nabla\cdot S^\perp-\widehat{b}\cdot\nabla S^\perp\cdot\widehat{b})n_{ij}\end{aligned} \tag{17}$$

where the derivatives act only on the heat flux components. This yields (the superscript (0) refers to equilibrium quantities)

$$\partial_t\Pi_{xx}-2\Omega\Pi_{xy}+p_\perp^{(0)}(\partial_x u_x-\partial_y u_y)+\frac{1}{2}(\partial_x S_x^\perp-\partial_y S_y^\perp)=0 \tag{18}$$

$$\partial_t\Pi_{xy}+2\Omega\Pi_{xx}+p_\perp^{(0)}(\partial_x u_y+\partial_y u_x)+\frac{1}{2}(\partial_y S_x^\perp+\partial_x S_y^\perp)=0 \tag{19}$$

$$\partial_t\Pi_{xz}-\Omega\Pi_{yz}+p_\perp^{(0)}\partial_x u_z+p_\parallel^{(0)}\partial_z u_x+\partial_x S_z^\perp+\partial_z S_x^\parallel-(p_\perp^{(0)}-p_\parallel^{(0)})\partial_t\widehat{b}_x=0 \tag{20}$$

$$\partial_t\Pi_{yz}+\Omega\Pi_{xz}+p_\perp^{(0)}\partial_y u_z+p_\parallel^{(0)}\partial_z u_y+\partial_y S_z^\perp+\partial_z S_y^\parallel-(p_\perp^{(0)}-p_\parallel^{(0)})\partial_t\widehat{b}_y=0 \tag{21}$$

together with $\Pi_{xx}=-\Pi_{yy}$ and $\Pi_{zz}=0$. Defining the transverse divergence of the gyroviscous stress tensor $\nabla_\perp\cdot\Pi_\perp$ as the vector of components $\Big(\partial_x\Pi_{xx}+\partial_y\Pi_{xy},\partial_x\Pi_{xy}+\partial_y\Pi_{yy},0\Big)$ and introducing the unit vector $\widehat{z}$ in the direction of the ambient field, Eqs. (18) and (19) then give

$$\nabla_\perp\cdot\Pi_\perp+\frac{1}{4\Omega}\Delta_\perp S^\perp\times\widehat{z}=-\frac{p_\perp^{(0)}}{2\Omega}\Delta_\perp u\times\widehat{z}-\frac{1}{2\Omega}\partial_t\left(\nabla_\perp\cdot\Pi_\perp\right)\times\widehat{z}. \tag{22}$$

On the other hand, defining the vector $\Pi_z=(\Pi_{xz},\Pi_{yz},\Pi_{zz}=0)$, Eqs. (20) and (21) rewrite

$$-\Omega\Pi_z\times\widehat{z}+\partial_z S_\perp^\parallel=-\nabla_\perp S_z^\perp-p_\perp^{(0)}\nabla_\perp u_z-p_\parallel^{(0)}\partial_z u_\perp+\Big(p_\perp^{(0)}-p_\parallel^{(0)}\Big)\partial_t\widehat{b}_\perp-\partial_t\Pi_z. \tag{23}$$

E. Dynamics of the heat flux vectors

Equation (4) for the heat flux tensor involves the divergence of the fourth order moment $\mathbf{r}$, that at this step should be simplified in order to conveniently close the hierarchy at the present order. We first note that instead of dealing with the fourth order moment $\mathbf{r}$, it is convenient to isolate the deviation from the product of second order moments by writing

$$\rho r_{ijkl} = P_{ij}P_{lk} + P_{ik}P_{jl} + P_{il}P_{jk} + P_{ij}\Pi_{lk} + P_{ik}\Pi_{jl} + P_{il}\Pi_{jk}$$
$$+\Pi_{ij}P_{lk} + \Pi_{ik}P_{jl} + \Pi_{il}P_{jk} + \rho\widetilde{r}_{ijkl}. \tag{24}$$

The correction term $\rho\widetilde{r}_{ijkl}$ a priori includes a contribution of the form $\Pi_{ij}\Pi_{lk} + \Pi_{ik}\Pi_{jl} + \Pi_{il}\Pi_{jk}$ that we here neglect since, as already mentioned, contributions from the gyroviscous stress are retained in linear terms only (except in Eqs. (8) and (9) in order to ensure energy conservation). This algebraic transformation allows significant simplifications in the forthcoming equations. Second, we make the approximation of retaining only the gyrotropic part of the tensor $\widetilde{\mathbf{r}}$ that is then given by

$$\widetilde{r}_{ijkl} = \frac{\widetilde{r}_{\|\|}}{3}(\tau_{ij}\tau_{kl} + \tau_{ik}\tau_{jl} + \tau_{il}\tau_{jk}) + \widetilde{r}_{\|\perp}(n_{ij}\tau_{kl} + n_{ik}\tau_{jl} + n_{il}\tau_{jk} \tag{25}$$

$$+\tau_{ij}n_{kl} + \tau_{ik}n_{jl} + \tau_{il}n_{jk}) + \frac{\widetilde{r}_{\perp\perp}}{2}(n_{ij}n_{kl} + n_{ik}n_{jl} + n_{il}n_{jk}). \tag{26}$$

The scalar quantities $r_{\|\|} = r_{ijlk}\tau_{ij}\tau_{kl}$, $r_{\perp\|} = \frac{1}{2}r_{ijlk}n_{ij}\tau_{kl}$ and $r_{\perp\perp} = \frac{1}{4}r_{ijlk}n_{ij}n_{kl}$ are related to $\widetilde{r}_{\|\|}$, $\widetilde{r}_{\|\perp}$ and $\widetilde{r}_{\perp\perp}$ (given by similar formulas with r_{ijkl} replaced by $\widetilde{r}_{ijkl}$) by

$$\widetilde{r}_{\|\|} = r_{\|\|} - 3\frac{p_\|^2}{\rho} \tag{27}$$

$$\widetilde{r}_{\|\perp} = r_{\|\perp} - \frac{p_\perp p_\|}{\rho} \tag{28}$$

$$\widetilde{r}_{\perp\perp} = r_{\perp\perp} - 2\frac{p_\perp^2}{\rho}. \tag{29}$$

One derives the equations for the heat flux vectors by writing $\frac{dS_i^\|}{dt} = S_{ijk}\frac{d\tau_{jk}}{dt} + \frac{dS_{ijk}}{dt}\tau_{jk}$ and $\frac{dS_i^\perp}{dt} = -S_{ijk}\frac{d\tau_{jk}}{dt} + \frac{dS_{ijk}}{dt}n_{jk}$. The first term in the above equations is given by

$$S_{ijk}\frac{d\tau_{jk}}{dt} = 2(S^\perp - S^\|)\cdot\widehat{b}\frac{d\widehat{b}_i}{dt} + 2S_j^\|\frac{d\tau_{ij}}{dt} \tag{30}$$

and the second terms are computed using the dynamical equation for the third order moment. One gets

$$\frac{dS_i^\perp}{dt} = -(S^\perp - S^\|)\cdot\widehat{b}\frac{d\widehat{b}_i}{dt} - S_j^\|\frac{d\tau_{ij}}{dt} - \frac{1}{2\rho}(P_{ij}P_{kl} + P_{ik}P_{jl} + P_{il}P_{jk}$$
$$+P_{ij}\Pi_{kl} + P_{ik}\Pi_{jl} + P_{il}\Pi_{jk} + \Pi_{ij}P_{kl} + \Pi_{ik}P_{jl} + \Pi_{il}P_{jk})\partial_l\tau_{jk}$$
$$-(P_{jl} + \Pi_{jl})\partial_l\Big(\frac{p^\perp}{\rho}(\tau_{ij} + 2n_{ij})\Big) - P_{jl}\partial_l\Big(\frac{1}{\rho}n_{jk}\Pi_{ik}\Big) + \frac{1}{\rho}\Pi_{ik}n_{jk}\partial_l\Pi_{jl}$$
$$-(S^\perp\cdot\nabla)u_i - (\nabla\cdot u)S_i^\perp - \frac{1}{2}\partial_l u_j\Big(S_i^\perp n_{jl} + S_m^\perp n_{mj}n_{il} + S_l^\perp n_{ij}$$
$$+S_m^\perp\tau_{mi}n_{jl} + S_m^\perp\tau_{ml}n_{ij} + 2S_k^\|\tau_{il}n_{jk}\Big) + \Omega\epsilon_{ijl}S_j^\perp\widehat{b}_l - \frac{1}{2}n_{jk}\partial_l\widetilde{r}_{ijkl} \tag{31}$$

and

$$\frac{dS_i^\|}{dt} = 2(S^\perp - S^\|)\cdot\widehat{b}\frac{d\widehat{b}_i}{dt} + 2S_j^\|\frac{d\tau_{ij}}{dt} + \frac{1}{\rho}(P_{ij}P_{kl} + P_{ik}P_{jl} + P_{il}P_{jk}$$
$$+P_{ij}\Pi_{kl} + P_{ik}\Pi_{jl} + P_{il}\Pi_{jk} + \Pi_{ij}P_{kl} + \Pi_{ik}P_{jl} + \Pi_{il}P_{jk})\partial_l\tau_{jk}$$

$$
\begin{aligned}
&-(P_{jl}+\Pi_{jl})\partial_l\Big(\frac{p_\parallel}{\rho}(n_{ij}+3\tau_{ij})\Big)-2P_{jl}\partial_l\Big(\frac{1}{\rho}\tau_{jk}\Pi_{ik}\Big)+\frac{2}{\rho}\Pi_{ik}\tau_{jk}\partial_l\Pi_{jl}\\
&-(S^\parallel\cdot\nabla)u_i-(\nabla\cdot u)S_i^\parallel-2\partial_l u_j\Big(S_k^\perp\tau_{jk}n_{il}+S_i^\parallel\tau_{jl}+S_l^\parallel\tau_{ij}-(S^\parallel\cdot\widehat{b})\tau_{il}\widehat{b}_j\Big)\\
&+\Omega\epsilon_{ijl}S_j^\parallel\widehat{b}_l-\tau_{jk}\partial_l\widetilde{r}_{ijkl}
\end{aligned}
\tag{32}
$$

which do not totally identify with the result of Ref. [15].

F. Second order approximation of the non-gyrotropic pressures and heat fluxes

Noting by inspection of Eqs. (31) and (32) that the magnitude of the transverse components of the heat flux vectors scales proportionally to the inverse gyrofrequency of the ions, we linearize the equations for these quantities, while we retain the nonlinear dynamics of the longitudinal components (see Section II G). Using $\partial_l(\widetilde{r}_{ixxl}+\widetilde{r}_{iyyl})=2\partial_i\widetilde{r}_{\perp\perp}$ for $i=x$ or y and $\partial_l\widetilde{r}_{izzl}=\partial_i\widetilde{r}_{\parallel\perp}$, and introducing the temperatures $T_\parallel=mp_\parallel/\rho$ and $T_\perp=mp_\perp/\rho$ where m is the mass of the considered particles, one has

$$
\frac{T_\perp^{(0)}}{m}\nabla_\perp\cdot\Pi_\perp-\Omega S_\perp^\perp\times\widehat{z}=-2\frac{p_\perp^{(0)}}{m}\nabla_\perp T_\perp^{(1)}-2\nabla_\perp\widetilde{r}_{\perp\perp}-\partial_t S_\perp^\perp. \tag{33}
$$

Similarly,

$$
2\frac{T_\parallel^{(0)}}{m}\partial_z\Pi_z-\Omega S_\perp^\parallel\times\widehat{z}=-\frac{p_\perp^{(0)}}{m}\nabla_\perp T_\parallel^{(1)}-2\frac{p_\parallel^{(0)}-p_\perp^{(0)}}{m}T_\parallel^{(0)}\partial_z\widehat{b}_\perp-\nabla_\perp\widetilde{r}_{\parallel\perp}-\partial_t S_\perp^\parallel. \tag{34}
$$

Combining Eqs. (22) and (33) and defining the square Larmor radius $r_L^2=\dfrac{T_\perp^{(0)}}{m\Omega^2}$ gives

$$
\begin{aligned}
\Big(1+\frac{1}{4}r_L^2\Delta_\perp\Big)\nabla_\perp\cdot\Pi_\perp&=\frac{p_\perp^{(0)}}{2\Omega}\widehat{z}\times\Delta_\perp u-\frac{\rho^{(0)}}{2m}r_L^2\Delta_\perp\nabla_\perp T_\perp^{(1)}\\
&\quad-\frac{1}{2\Omega^2}\Delta_\perp\nabla_\perp\widetilde{r}_{\perp\perp}-\frac{1}{2\Omega}\partial_t\Big(\nabla_\perp\cdot\Pi_\perp\times\widehat{z}+\frac{1}{2\Omega}\Delta_\perp S_\perp^\perp\Big)
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
\Big(1+\frac{1}{4}r_L^2\Delta_\perp\Big)S_\perp^\perp&=\Big(\frac{2p_\perp^{(0)}}{m\Omega}\widehat{z}\times\nabla_\perp T_\perp^{(1)}-\frac{p_\perp^{(0)}}{2}r_L^2\Delta_\perp u_\perp+\frac{2}{\Omega}\widehat{z}\times\nabla_\perp\widetilde{r}_{\perp\perp}\Big)\\
&\quad-\partial_t\Big(\frac{r_L^2}{2}\nabla_\perp\cdot\Pi_\perp-\frac{1}{\Omega}\widehat{z}\times S^\perp\Big).
\end{aligned}
\tag{36}
$$

Similarly, combining Eqs. (23) and (34) gives

$$
\begin{aligned}
\Big(1+2\frac{T_\parallel^{(0)}}{m\Omega^2}\partial_{zz}\Big)\Pi_z&=\frac{\widehat{z}}{\Omega}\times\Big(\nabla_\perp S_z^\perp+p_\perp^{(0)}\nabla_\perp u_z+p_\parallel^{(0)}\partial_z u_\perp-(p_\perp^{(0)}-p_\parallel^{(0)})\partial_t\widehat{b}_\perp+\partial_t\Pi_z\Big)\\
&\quad-\frac{1}{\Omega^2}\partial_z\Big(\frac{p_\perp^{(0)}}{m}\nabla_\perp T_\parallel^{(1)}-2\frac{p_\perp^{(0)}-p_\parallel^{(0)}}{m}T_\parallel^{(0)}\partial_z\widehat{b}_\perp+\nabla_\perp\widetilde{r}_{\parallel\perp}+\partial_t S_\perp^\parallel\Big)
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\Big(1+2\frac{T_\parallel^{(0)}}{m\Omega^2}\partial_{zz}\Big)S_\perp^\parallel&=\frac{\widehat{z}}{\Omega}\times\Big(\frac{p_\perp^{(0)}}{m}\nabla_\perp T_\parallel^{(1)}-2\frac{p_\perp^{(0)}-p_\parallel^{(0)}}{m}T_\parallel^{(0)}\partial_z\widehat{b}_\perp+\nabla_\perp\widetilde{r}_{\parallel\perp}+\partial_t S_\perp^\parallel\Big)\\
&\quad-\frac{2T_\parallel^{(0)}}{m\Omega^2}\partial_z\Big(\nabla_\perp S_z^\perp+p_\perp^{(0)}\nabla_\perp u_z+p_\parallel^{(0)}\partial_z u_\perp-\Big(p_\perp^{(0)}-p_\parallel^{(0)}\Big)\partial_t\widehat{b}_\perp-\partial_t\Pi_z\Big).
\end{aligned}
\tag{38}
$$

Note that the operators in the l.h.s. of eqs. (35)-(38) cannot be inverted for any wavenumber, indicating the limitation of the fluid approach to large scales, both in the longitudinal and transverse directions. At second order in terms of $\dfrac{\omega}{\Omega}\sim\sqrt{\dfrac{2T_\parallel^{(0)}}{m}}\dfrac{k_z}{\Omega}\sim r_L k_\perp$, these equations simplify into

$$
\nabla_\perp\cdot\Pi_\perp=\frac{p_\perp^{(0)}}{2\Omega}\widehat{z}\times\Delta_\perp u-\frac{\rho^{(0)}}{2m}r_L^2\Delta_\perp\nabla_\perp T_\perp^{(1)}-\frac{1}{2\Omega^2}\Delta_\perp\nabla_\perp\widetilde{r}_{\perp\perp}+\frac{1}{2\Omega}\widehat{z}\times\partial_t\nabla_\perp\cdot\Pi_\perp \tag{39}
$$

$$S_\perp^\perp = \frac{2p_\perp^{(0)}}{m\Omega}\widehat{z}\times\nabla_\perp T_\perp^{(1)} - \frac{p_\perp^{(0)}}{2} r_L^2 \Delta_\perp u_\perp + \frac{2}{\Omega}\widehat{z}\times\nabla_\perp \widetilde{r}_{\perp\perp} + \frac{1}{\Omega}\widehat{z}\times\partial_t S^\perp \tag{40}$$

$$\Pi_z = \frac{\widehat{z}}{\Omega}\times\Big(\nabla_\perp S_z^\perp + p_\perp^{(0)}\nabla_\perp u_z + p_\parallel^{(0)}\partial_z u_\perp - (p_\perp^{(0)} - p_\parallel^{(0)})\partial_t \widehat{b}_\perp + \partial_t \Pi_z\Big)$$
$$-\frac{1}{\Omega^2}\partial_z\Big(\frac{p_\perp^{(0)}}{m}\nabla_\perp T_\parallel^{(1)} - 2\frac{p_\perp^{(0)} - p_\parallel^{(0)}}{m} T_\parallel^{(0)}\partial_z\widehat{b}_\perp + \nabla_\perp\widetilde{r}_{\parallel\perp}\Big) \tag{41}$$

$$S_\perp^\parallel = \frac{\widehat{z}}{\Omega}\times\Big(\frac{p_\perp^{(0)}}{m}\nabla_\perp T_\parallel^{(1)} - 2\frac{p_\perp^{(0)} - p_\parallel^{(0)}}{m} T_\parallel^{(0)}\partial_z\widehat{b}_\perp + \nabla_\perp\widetilde{r}_{\parallel\perp} + \partial_t S_\perp^\parallel\Big)$$
$$-\frac{2T_\parallel^{(0)}}{m\Omega^2}\partial_z\Big(\nabla_\perp S_z^\perp + p_\perp^{(0)}\nabla_\perp u_z + p_\parallel^{(0)}\partial_z u_\perp - (p_\perp^{(0)} - p_\parallel^{(0)})\partial_t\widehat{b}_\perp\Big). \tag{42}$$

The last term in the r.h.s. of Eq. (39) can be consistently replaced by $\dfrac{-1}{4\Omega^2}p_\perp^{(0)}\Delta_\perp\partial_t u_\perp$, that in (40) by $\dfrac{-2p_\perp^{(0)}T_\perp^{(0)}}{m\Omega^2}\nabla_\perp\partial_t\Big(\dfrac{T_\perp}{T_\perp^{(0)}}\Big)$. A similar substitution is made in Eqs. (41) and (42), the terms involving $\partial_t\Pi_z$ and $\partial_t S_\perp^\parallel$, being replaced by their leading order expressions within the linear description.

G. Simplified nonlinear equations for the longitudinal components of the heat flux vectors

In deriving the dynamical equations governing the longitudinal components of the heat flux vectors, we retain the coupling to the transverse components and to the gyroviscous tensor at the linear level only, because of the presence of a $1/\Omega$ factor, and the assumption that the present equations are restricted to the description of the large scales. We retain the other couplings that include quadratic contributions with respect to the fluctuations (weak nonlinearity regime). Note that the variation of $\widehat{b}_z$ has a magnitude that scales like the square of the perturbations. One then gets

$$\partial_t S_z^\parallel + \nabla\cdot(S_z^\parallel u) + 3S_z^\parallel\partial_z u_z + 3p_\parallel(\widehat{b}\cdot\nabla)\Big(\frac{p_\parallel}{\rho}\Big) - p_\perp\widehat{b}_\perp\cdot\nabla_\perp\Big(\frac{p_\parallel}{\rho}\Big)$$
$$+\frac{2p_\parallel}{\rho}(p_\parallel - p_\perp)\partial_z\widehat{b}_z + \nabla\cdot(\widetilde{r}_{\parallel\parallel}\widehat{b}) - 3\widetilde{r}_{\parallel\perp}\nabla\cdot\widehat{b} - (b_\perp\cdot\nabla_\perp)\widetilde{r}_{\parallel\perp} = 0. \tag{43}$$

Similarly, when considering the equation governing $S_z^\perp$, one gets

$$\partial_t S_z^\perp + \nabla\cdot(uS_z^\perp) + S_z^\perp\nabla\cdot u + p_\parallel(\widehat{b}\cdot\nabla)\Big(\frac{p_\perp}{\rho}\Big) - 2p_\perp(\widehat{b}_\perp\cdot\nabla_\perp)\Big(\frac{p_\perp}{\rho}\Big)$$
$$+\frac{p_\perp}{\rho}\Big(\partial_x\Pi_{xz} + \partial_y\Pi_{yz}\Big) + \nabla\cdot(\widetilde{r}_{\parallel\perp}\widehat{b}) + \Big(\frac{p_\perp(p_\parallel - p_\perp)}{\rho} - \widetilde{r}_{\perp\perp} + \widetilde{r}_{\parallel\perp}\Big)(\nabla\cdot\widehat{b})$$
$$-(\widehat{b}_\perp\cdot\nabla_\perp)\widetilde{r}_{\perp\perp} = 0. \tag{44}$$

III. LINEAR KINETIC THEORY

Let us assume that the equilibrium state is characterized for each particle species by a bi-Maxwellian distribution function $f_0 = \dfrac{1}{(2\pi)^{3/2}}\dfrac{m^{3/2}}{T_\perp^{(0)}T_\parallel^{(0)1/2}}\exp\Big\{-\Big(\dfrac{m}{2T_\parallel^{(0)}}v_\parallel^2 + \dfrac{m}{2T_\perp^{(0)}}v_\perp^2\Big)\Big\}$. For small disturbances, the perturbation f_1 of the distribution function is linearly expressed in terms of the parallel and transverse electric field components that are conveniently written in terms of potentials, in the form $E_z = -\partial_z\Psi$ and $E_\perp = -\nabla_\perp\Phi - \dfrac{1}{c}\partial_t A_\perp$ with $B = B_0\hat{z} + \nabla\times A$ and the gauge condition $\nabla\cdot A = 0$. We also denote by b_z the magnetic field fluctuations along the z-direction.

The hydrodynamic moments are easily computed in a low frequency expansion, retaining only contributions up to order $\dfrac{\omega}{\Omega}\sim\dfrac{k_z}{\Omega}\sqrt{\dfrac{2T_\perp^{(0)}}{m}}\ll 1$, with no condition on $\dfrac{k_\perp}{\Omega}\sqrt{\dfrac{2T_\perp^{(0)}}{m}}$. Let us also introduce $b = \dfrac{T_\perp^{(0)}k_\perp^2}{m\Omega^2} = k_\perp^2 r_L^2$, $\zeta = \dfrac{\omega}{|k_z|}\sqrt{\dfrac{m}{2T_\parallel^{(0)}}}$ and define the functions $\Gamma_\nu(b) = e^{-b}I_\nu(b)$ in terms of the modified Bessel function $I_\nu(b)$. A standard calculation leads to the following results in terms of the plasma response function $R(\zeta) = 1 + \zeta Z(\zeta)$, where $Z(\zeta)$ is the plasma dispersion function.

The longitudinal and transverse temperature perturbations $T_\parallel^{(1)}$ and $T_\perp^{(1)}$ are given by

$$\frac{T_\parallel^{(1)}}{T_\parallel^{(0)}} = \Big(1 - R(\zeta) + 2\zeta^2 R(\zeta)\Big) \frac{T_\perp^{(0)}}{T_\parallel^{(0)}} \Big[\Big(\Gamma_1(b) - \Gamma_0(b)\Big)\frac{b_z}{B_0} - \Gamma_0(b)\frac{e\Psi}{T_\perp^{(0)}}\Big] \tag{45}$$

and

$$\frac{T_\perp^{(1)}}{T_\perp^{(0)}} = \Big(\frac{T_\perp^{(0)}}{T_\parallel^{(0)}} R(\zeta) - 1\Big)\Big(- 2b\Gamma_1(b) + 2b\Gamma_0(b) - \Gamma_0(b)\Big)\frac{b_z}{B_0} - \Big(b\Gamma_1(b) - b\Gamma_0(b)\Big) R(\zeta)\frac{e\Psi}{T_\parallel^{(0)}}$$
$$+\Big(b\Gamma_1(b) - b\Gamma_0(b)\Big)\frac{e}{T_\perp^{(0)}}\Big(\Phi + \frac{k_z^2}{k_\perp^2}(\Phi - \Psi)\Big). \tag{46}$$

When restricted to the linear approximation, the elements of the heat flux tensor reduce to $q_{ijk} = n^{(0)} m \int v_i v_j v_k f_1 d^3v - u_i p_{jk}^{(0)} - u_j p_{ik}^{(0)} - u_k p_{ij}^{(0)}$. For the flux vectors $S_i^\parallel = q_{ijk}\widehat{b}_j\widehat{b}_k$ and $S_i^\perp = \frac{1}{2} q_{ijk}(\delta_{jk} - \widehat{b}_j\widehat{b}_k)$, one then has $S_i^\parallel = n^{(0)} m \int v_i v_\parallel^2 f_1 d^3v - p_\parallel^{(0)}(u_i + 2\delta_{i3}u_z)$ and $S_i^\perp = \frac{n^{(0)}m}{2}\int v_i v_\perp^2 f_1 d^3v - p_\perp^{(0)}(2u_i - \delta_{i3}u_z)$.

It results that

$$S_z^\parallel = -p_\parallel^{(0)} \frac{T_\perp^{(0)}}{T_\parallel^{(0)}}\frac{\omega}{k_z}\Big(1 - 3R(\zeta) + 2\zeta^2 R(\zeta)\Big)\Big[\Big(\Gamma_0(b) - \Gamma_1(b)\Big)\frac{b_z}{B_0} + \Gamma_0(b)\frac{e\Psi}{T_\perp^{(0)}}\Big] \tag{47}$$

and

$$S_z^\perp = p_\perp^{(0)}\Big\{\frac{T_\perp^{(0)}}{T_\parallel^{(0)}}\frac{\omega}{k_z}\Big(2b\Gamma_0(b) - \Gamma_0(b) - 2b\Gamma_1(b)\Big)R(\zeta)\frac{b_z}{B_0} + \frac{\omega}{k_z} b\Big(\Gamma_0(b) - \Gamma_1(b)\Big)R(\zeta)\frac{e\Psi}{T_\parallel^{(0)}}$$
$$-\frac{T_\perp^{(0)} - T_\parallel^{(0)}}{m}\frac{k_z}{\omega}\Big(b\Gamma_0(b) - b\Gamma_1(b)\Big)\frac{e}{T_\perp^{(0)}}\Big(1 + \frac{k_z^2}{k_\perp^2}\Big)(\Phi - \Psi)\Big\}. \tag{48}$$

For a gyrotropic equilibrium distribution function, symmetric in the direction of the ambient field, the elements of the fourth order moment perturbation read $r_{ijkl}^{(1)} = n^{(0)} m \int v_i v_j v_k v_l f_1 d^3v$. One computes the scalar quantities $r_{\parallel\parallel}^{(1)} = r_{ijlk}^{(1)}\widehat{b}_i\widehat{b}_j\widehat{b}_k\widehat{b}_l = n^{(0)} m \int v_\parallel^4 f_1 d^3v$, $r_{\parallel\perp}^{(1)} = \frac{1}{2} r_{ijlk}^{(1)}(\delta_{ij} - \widehat{b}_i\widehat{b}_j)\widehat{b}_k\widehat{b}_l = \frac{1}{2} n^{(0)} m \int v_\parallel^2 v_\perp^2 f_1 d^3v$ and $r_{\perp\perp}^{(1)} = \frac{1}{4} r_{ijlk}^{(1)}(\delta_{ij} - \widehat{b}_i\widehat{b}_j)(\delta_{lk} - \widehat{b}_k\widehat{b}_l) = \frac{1}{4} n^{(0)} m \int v_\perp^4 f_1 d^3v$. After linearization of Eqs. (27)–(29) one gets

$$\widetilde{r}_{\parallel\parallel} = \frac{p_\parallel^{(0)} T_\perp^{(0)}}{m}\Big[2\zeta^2\Big(1 + 2\zeta^2 R(\zeta)\Big) + 3\Big(R(\zeta) - 1\Big) - 12\zeta^2 R(\zeta)\Big]\Big[\Big(\Gamma_1(b) - \Gamma_0(b)\Big)\frac{b_z}{B_0} - \Gamma_0(b)\frac{e\Psi}{T_\perp^{(0)}}\Big] \tag{49}$$

$$\widetilde{r}_{\parallel\perp} = \frac{{p_\perp^{(0)}}^2}{\rho^{(0)}}\Big(1 - R(\zeta) + 2\zeta^2 R(\zeta)\Big)\Big[\Big(2b\Gamma_0(b) - \Gamma_0(b) - 2b\Gamma_1(b)\Big)\frac{b_z}{B_0} + b\Big(\Gamma_0(b) - \Gamma_1(b)\Big)\frac{e\Psi}{T_\perp^{(0)}}\Big] \tag{50}$$

$$\widetilde{r}_{\perp\perp} = \frac{{p_\perp^{(0)}}^2}{\rho^{(0)}}\Big\{\Big(4b^4\Gamma_1(b) - 4b^2\Gamma_0(b) - b\Gamma_1(b) + 3b\Gamma_0(b)\Big)\Big(\frac{T_\perp^{(0)}}{T_\parallel^{(0)}}R(\zeta) - 1\Big)\frac{b_z}{B_0}$$
$$+\Big(2b^2\Gamma_1(b) + b\Gamma_1(b) - 2b^2\Gamma_0(b)\Big)R(\zeta)\frac{e\Psi}{T_\parallel^{(0)}} + \Big(2b^2\Gamma_0(b) - 6b\Gamma_1(b)\Big)\frac{e}{T_\perp^{(0)}}\Big(\Phi + \frac{k_z^2}{k_\perp^2}(\Phi - \Psi)\Big)\Big\}. \tag{51}$$

IV. A LANDAU FLUID CLOSURE

When comparing the expression of $\widetilde{r}_{\|\|\|}^{(1)}$ with those of $S_z^{\|}$ or $T_{\|}^{(1)}$ provided by the kinetic theory, one gets

$$\widetilde{r}_{\|\|\|} = \sqrt{\frac{2T_{\|}^{(0)}}{m}} \frac{2\zeta^2\Big(1+2\zeta^2 R(\zeta)\Big) + 3\Big(R(\zeta)-1\Big) - 12\zeta^2 R(\zeta)}{2\zeta \mathrm{sgn}\,(k_z)\Big(1-3R(\zeta)+2\zeta^2 R(\zeta)\Big)} S_z^{\|} \equiv \sqrt{\frac{2T_{\|}^{(0)}}{m}} \mathcal{F}_S S_z^{\|}. \tag{52}$$

and

$$\widetilde{r}_{\|\|\|} = \frac{p_{\|}^{(0)}T_{\|}^{(0)}}{m}\, \frac{2\zeta^2\Big(1+2\zeta^2 R(\zeta)\Big) + 3\Big(R(\zeta)-1\Big) - 12\zeta^2 R(\zeta)}{1-R(\zeta)+2\zeta^2 R(\zeta)}\, \frac{T_{\|}^{(1)}}{T_{\|}^{(0)}} \equiv \frac{p_{\|}^{(0)}T_{\|}^{(0)}}{m} \mathcal{F}_T \frac{T_{\|}^{(1)}}{T_{\|}^{(0)}}. \tag{53}$$

One then notices that when replacing the plasma response function R by its four pole approximant

$$R_4(\zeta) = \frac{4 - 2\,i\sqrt{\pi}\zeta + (8-3\,\pi)\,\zeta^2}{4 - 6\,i\sqrt{\pi}\zeta + (16-9\,\pi)\,\zeta^2 + 4\,i\sqrt{\pi}\zeta^3 + (6\,\pi - 16)\,\zeta^4},$$

one has the identity

$$\lambda \frac{\mathcal{F}_S}{\mathcal{F}_T} + i\mu \frac{k_z}{|k_z|} = \mathcal{F}_S \tag{54}$$

with $\lambda = \dfrac{32-9\pi}{3\pi-8}$ and $\mu = \dfrac{-2\sqrt{\pi}}{3\pi-8}$. This leads to the closure relation

$$\widetilde{r}_{\|\|\|} = \lambda p_{\|}^{(0)} \frac{T_{\|}^{(0)}}{m} \frac{T_{\|}^{(1)}}{T_{\|}^{(0)}} + \mu \sqrt{\frac{2T_{\|}^{(0)}}{m}} \frac{ik_z}{|k_z|} S_z^{\|}, \tag{55}$$

which identifies with Eq. (34) of Ref. [8]. Note that this closure is here established with no assumption on the magnitude of the transverse wavenumbers.

On the other hand, $\widetilde{r}_{\|\perp}$ can be expressed in terms of $S_z^{\perp}$ and the parallel current j_z. One has

$$\widetilde{r}_{\|\perp} = \sqrt{\frac{2T_{\|}^{(0)}}{m}} \frac{1-R(\zeta)+2\zeta^2 R(\zeta)}{2\zeta R(\zeta)} \Big[S_z^{\perp} + \Big(\Gamma_0(b) - \Gamma_1(b)\Big) \frac{p_{\perp}^{(0)} p_{\|}^{(0)}}{\rho^{(0)} v_A^2} \Big(\frac{T_{\perp}^{(0)}}{T_{\|}^{(0)}} - 1\Big) \frac{j_z}{en^{(0)}}\Big] \tag{56}$$

where $v_A = B_0/\sqrt{4\pi\rho^{(0)}}$ is the Alfvén velocity and $\rho^{(0)}$ the plasma density at equilibrium.

When dealing with $\widetilde{r}_{\|\perp}$, the approximation consisting in replacing the plasma response function R by its two pole Padé approximant $R_2(\zeta) = 1/(1 - i\sqrt{\pi}\zeta - 2\zeta^2)$, as performed to obtain Eq. (35) of Ref. [8] is not satisfactory since it does not correctly reproduce the large ζ decay of the imaginary part of the fraction $\dfrac{1-R(\zeta)+2\zeta^2 R(\zeta)}{2\zeta R(\zeta)}$. Similar possible overestimate of the Landau damping by Landau fluid models are mentioned in Ref. [16]. In contrast, using $R_3(\zeta) = \dfrac{2 - i\sqrt{\pi}\zeta}{2 - 3i\sqrt{\pi}\zeta - 4\zeta^2 + 2i\sqrt{\pi}\zeta^3}$, one has the approximation $\dfrac{1-R(\zeta)+2\zeta^2 R(\zeta)}{2\zeta R(\zeta)} \approx \dfrac{i\sqrt{\pi}}{-2+i\sqrt{\pi}\zeta}$. This leads to write the evolution equation

$$\Big(\frac{d}{dt} - \frac{2}{\sqrt{\pi}} \sqrt{\frac{2T_{\|}^{(0)}}{m}} \mathcal{H}_z \partial_z\Big) \widetilde{r}_{\|\perp} + \frac{2T_{\|}^{(0)}}{m} \partial_z \Big[S_z^{\perp} + \frac{p_{\perp}^{(0)}}{v_A^2} \Big(\frac{T_{\perp}^{(0)} - T_{\|}^{(0)}}{m_p}\Big) \frac{j_z}{en^{(0)}}\Big] = 0, \tag{57}$$

where in the large-scale limit we are here concerned with, we made the expansion $b\Gamma_0(b) - b\Gamma_1(b) \approx b = k_{\perp}^2 r_L^2$. The notation m_p is used in situations where the proton mass is to remain unchanged when turning to the corresponding equation for the electron. In Fourier space, the Hilbert transform $\mathcal{H}_z$ reduces to the multiplication by $i\,\mathrm{sgn}\,k_z$. The convective derivative has been reintroduced to ensure Galilean invariance.

Finally, the reduced moment $\widetilde{r}_{\perp\perp}$ turns out to be totally negligible at large scales and will thus not be retained.

V. COMMENTS ON THE RESULTING MODEL

The equations derived above for the ions are easily adapted to the electrons for which they greatly simplify when making the approximation $m_e/m_p \ll 1$. This leads to neglect the non gyrotropic components of the corresponding pressure tensor. Note that the transverse components of the electron heat flux vectors survive due to the contributions of terms involving the product $m_e\Omega_e$ (see Section II.F). The system is to be supplemented by Faraday equation and Ampère's law where the displacement current is neglected. In this two-fluid formulation, energy is conserved, as discussed by Ramos.[14] It might nevertheless be advantageous to filter out the scales associated with electrostatic waves by prescribing quasi-neutrality, replacing the electron momentum equation by a generalized Ohm's law, and turning to a one-fluid description. Numerical simulations of a monofluid model obtained from a simplified version of the present model have shown that energy is in practice very well conserved.[10]

When compared with the previous model[12] designed to reproduce the oblique Alfvén wave dynamics, the present approach proves to be more systematic and, as discussed below, allows one to accurately simulate all dispersive MHD waves, including oblique and transverse magnetosonic waves (see Section VI). The previous model has on the other hand the advantage of including a nonlinear description of the gyroviscous tensor. It is of interest to see how, when linearized and restricted to the case of the Alfvén wave scaling (also neglecting the gyroviscous tensor contribution), the equations governing the gyrotropic heat fluxes in the present model compare with those of the previous one. It turns out that Ref. [12] unfortunately includes a few algebraic errors originating from a sign error leading to an incorrect factor 3 in Eq. (C.8), a missing multiplicative factor m_p/m_r in the r.h.s. of Eqs. (C.9) and (C.10) and a missing minus sign in front of the first occurrence of Ω_p/Ω_r in Eq. (C.12). This in particular affects the equations for the gyrotropic heat fluxes where the contribution $v_{\Delta e}^2$ in the r.h.s. of Eq. (56) should be suppressed, the square bracket in Eq. (66) replaced by $[v_{\Delta r}^2 \mathrm{sgn}\, q_r - v_A^2(\delta_{rp} - 1) - v_{th,r}^2 \delta_{rp}]/v_A^2$ and the factor 3 in the last term in the r.h.s. of Eq. (67) also discarded. After correcting these errors and taking into account that pressure and heat flux tensors were computed using barycentric velocities, one easily checks that the parallel heat flux equation is exactly recovered and that the equations for the perpendicular heat flux of both models identify in the isothermal limit where the time derivatives are negligible. This limitation originates from the insufficient order of the Padé approximant used in the previous model.

VI. MHD WAVE DYNAMICS

When restricted to a one or quasi one-dimensional dynamics along the ambient field, only the longitudinal components of the parallel and transverse heat flux vectors (that correspond to the gyrotropic contributions to the heat flux tensor) arise in the equations of motion. A long-wave reductive perturbative expansion performed on the resulting Landau-fluid model reproduces the kinetic derivative nonlinear Schrödinger equation derived from the VM equations for Alfvén waves with a typical length scale large compared with the ion Larmor radius,[17] up to the replacement of the plasma response function by appropriate Padé approximants. As a consequence, the modulational type instabilities (including filamentation[18]) of Alfvén waves and their weakly nonlinear developments are correctly reproduced.[9] Numerical simulations of such regimes are presented in Ref. [10] where a study of the decay instability is also presented and validated by comparison with hybrid simulations.[19]

As stressed in Ref. [13], the correct determination of the dispersion relation for transversally propagating magnetosonic waves requires a detailed description of non-gyrotropic contributions to the pressure and heat flux tensors. When restricted to a purely transverse dynamics, the present model reduces to the fluid model used in Ref. [13] that exactly reproduces the large-scale kinetic theory (note that a factor 3/2 is missing in front of the z-term in ϵ_{xy} given in Eq. (2.8) of the latter reference).

The present model easily reproduces the dispersion relation for kinetic Alfvén waves (KAW) for which the crucial ingredient is the contribution to the transverse velocity originating from the time derivative of the leading order gyroviscous stress [last term in Eq. (39)].[12,20,21] Whereas these KAWs are also captured by a low frequency expansion of the kinetic equations,[22,23] this is not the case for oblique Alfvén waves. The reason is that an expansion at order ω/Ω includes contributions of order $k_\perp^2 r_L^2$ when $k_z/k_\perp$ scales like $k_\perp r_L$ as for KAWs, but only includes terms of order $k_\perp r_L$ for finite angles of propagation. The same limitation holds for the gyrokinetic formalism. The present fluid formalism however allows one to obtain the correct linear dynamics for oblique Alfvén waves, as was shown in Ref. [21], using a Landau fluid model actually contained in the present one.

VII. CONCLUDING REMARKS

We have constructed a Landau fluid model that reproduces all large-scale dispersive MHD waves in a warm collisionless plasma. This model may be most useful not only for numerical simulations involving a broad range of scales, but also for analytic purposes, such as the computation of secondary instabilities. An example is provided by the filamentation instability of parallel propagating Alfvén waves. This mechanism may be relevant in the understanding of the evolution of Alfvén waves in magnetospheric plasmas that often display very filamentary structures.[24] The present model allows one to account for linear Landau damping, dominant FLR corrections as well as drift velocities, that play an important role in these plasmas whose equilibrium state often involves a large scale longitudinal current. The importance of nonlinear kinetic effects such as particle trapping that are here neglected should be estimated by comparison with fully kinetic simulations.

In a sufficiently anisotropic plasma, the mirror instability can develop, whose threshold is accurately reproduced by the present fluid model.[8,10] A difficulty nevertheless originates in that, for large-scale mirror modes, the growth rate of perturbations propagating in the most unstable direction scales like the transverse wave number of the perturbation, which makes the smallest scales retained in a large-scale simulation to be the most unstable. The instability actually reaches a maximal rate at a scale comparable to the ion Larmor radius and is arrested at smaller scales, under the effect of FLR corrections.[25] Small transverse scales are thus to be retained. A promising approach consists in expressing, at the level of the linear kinetic theory, non-gyrotropic contributions in a closed form suitable for being incorporated into fluid equations. Explicit reference to the plasma response function should in particular be eliminated. A model that reproduces the arrest of the mirror instability and that is simple enough to permit accurate numerical simulations will be presented in a forthcoming paper.[26]

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