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THE MULTI-WATER-BAG EQUATIONS FOR COLLISIONLESS KINETIC MODELING

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Abstract. In this paper we consider the multi-water-bag model for collisionless kinetic equations. The multi-water-bag representation of the statistical distribution function of particles can be viewed as a special class of exact weak solution of the Vlasov equation, allowing to reduce this latter into a set of hydrodynamic equations while keeping its kinetic character. After recalling the link of the multi-water-bag model with kinetic formulation of conservation laws, we derive different multi-water-bag (MWB) models, namely the Poisson-MWB, the quasineutral-MWB and the electromagnetic-MWB models. These models are very promising because they reveal to be very useful for the theory and numerical simulations of laser-plasma and gyrokinetic physics. In this paper we prove some existence and uniqueness results for classical solutions of these different models. We next propose numerical schemes based on Discontinuous Garlerkin methods to solve these equations. We then present some numerical simulations of non linear problems arising in plasma physics for which we know some analytical results.

1. Introduction.

Vlasov equation is a difficult one mainly because of its high dimensionality. For each particle species the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is defined in a 6D phase space. Even the simplest (one spatial dimension, one velocity dimension) implies a 2D phase space. Can it be reduced to the sole configuration space as in usual hydrodynamics ? In that last case the presence of collisions with frequency much greater than the inverse of all characteristic times implies the existence of a local thermodynamic equilibrium characterised by a density $n(\mathbf{r}, t)$, an average

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velocity $\mathbf{u}(\mathbf{r},t)$ and a temperature $T(\mathbf{r},t)$. A priori in a plasma the distribution function $f(\mathbf{r},\mathbf{v},t)$ is an arbitrary function of $\mathbf{r}$ and $\mathbf{v}$ (and t of course) and phase space is unavoidable.

An alternative approach is based on a water bag representation of the distribution function which is not an approximation but rather a special class of initial conditions. Introduced initially by DePackh [27], Hohl, Feix and Bertrand [31, 11, 12] the water bag model was shown to bring the bridge between fluid and kinetic description of a collisionless plasma, allowing to keep the kinetic aspect of the problem with the same complexity as the fluid model. Twenty years later, mathematicians have rediscovered this property using the kinetic formulation of scalar conservation laws. It was established in [20, 21, 22, 36] that scalar conservation laws can be lifted as linear hyperbolic equations by introducing an extra variable $\xi \in \mathbb{R}$ which can be interpreted as a scalar momentum or velocity variable. In ref [22] the Author proposed a numerical scheme, known as the transport-collapse method to solve this linear kinetic equation and has proved, using BV estimates and Kruzhkov type analysis, that this numerical solution converges to the entropy solution of scalar conservation laws. This result was also shown in [62] using averaging lemmas [37, 38, 29, 19] without bounded variation estimate. Soon after, it was shown in refs. [58, 52] that, without any approximations, entropy solutions of scalar conservation laws could be directly formulated in kinetic style, known as kinetic formulation. Its generalization to systems of conservation laws seems impossible except for very peculiar systems ([23, 53, 63, 4, 5, 6]) where the kinetic formulation of multibranch entropy solutions have been developped. One of those system is the isentropic gas dynamics system with $\gamma = 3$ for which, long time ago, the link with the Vlasov kinetic equation was pointed out in [11] as the so called water bag model. Let us notice that the multibranch entropy solutions have been used for multivalued geometric optics computations and multiphase computations of the semiclassical limit of the Schrödinger equation [40, 41, 48, 42].

This paper deals with three different MWB models. The first one is the Poisson-multi-water-bag model which corresponds to a special class of weak solution of the Vlasov-Poisson system and thus constitutes a basic model in kinetic collisionless equations by which we must start. The second model is the quasineutral-multi-water-bag model where the coupling between waves and particles is obtained by equating the electrical potential to the particle density. This system is very fruitful because it represents the parallel dynamic of particles subjected to a strong magnetic field as it occurs in magnetic controlled fusion devices (tokamak) where gyrokinetic turbulence governs the energy confinement time [54, 55, 15, 18]. The third model is the electromagnetic-multi-water-bag model which is very useful in laser-plasma interaction because it provides a physical explanation for the formation of low frequency nonlinear coherent structures which are stable in long time, the so-called KEEN (Kinetic Electron Electrostatic Nonlinear) waves [2, 1, 34, 16]. These modes which have been observed in several simulations [2, 1, 34, 16] can be viewed as a non-steady variant of the well-known Bernstein-Greene-Kruskal (BGK)[13] modes that describe invariant traveling electrostatic waves in plasmas.

In order to introduce the water-bag model let us consider a 1D plasma (2D phase space (z,v)) in which at initial time the situation is as depicted in Fig. 1. Between the two curves v^+ and v^- we impose $f(t,z,v) = \mathcal{A}$ ($\mathcal{A}$ is a constant). For velocities bigger than v^+ and smaller than v^- we have $f(t,z,v) = 0$.

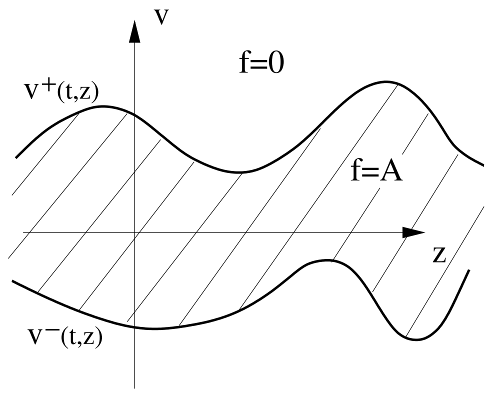


FIGURE 1. *The water bag model in phase space.*

According to phase space conservation property of the Vlasov equation, as long as v^+ and v^- remain single valued function, $f(t,z,v)$ remains equal to A for values of v such that $v^-(t,z) < v < v^+(t,z)$. Therefore the problem is entirely described by the two functions v^+ and v^-. Since a hydrodynamic description involves n, u and P (respectively density, average velocity and pressure) we can predict the possibility of casting the water bag model into the hydrodynamic frame with, in addition, an automatically provided state equation.

Remembering that a particle on the contour v^+ (or v^-) remains on this contour the equations for v^+ and v^- are (for instance for an electron plasma , E being the electric field and q the electric charge)

$$D_t v^{\pm}(t,z) = \partial_t v^{\pm}(t,z) + \left(v^{\pm}\partial_z v^{\pm}\right)(t,z) = \frac{q}{m}E(t,z). \tag{1}$$

We now introduce the density $n(t,z) = \mathcal{A}(v^+ - v^-)$ and the average (fluid) velocity $u(t,z) = \frac{1}{2}(v^+ + v^-)$ into equations (1) by adding and subtracting these two equations. We obtain

$$\begin{aligned} \partial_t n + \partial_z(nu) &= 0, & (2)\\ \partial_t u + u\partial_z u &= -\frac{1}{mn}\partial_z P + \frac{q}{m}E, & (3)\\ Pn^{-3} &= \frac{m}{12\,\mathcal{A}^2}. & (4) \end{aligned}$$

The equations (2)-(3)-(4) are respectively the continuity, Euler and state equation. This hydrodynamic description of the water bag model was pointed out for the first time by Bertrand and Feix [11] but the state equation (4) describes an invariant both in space and time while in the hydrodynamic model we obtain $D_t(Pn^{-\gamma}) = 0$. It must be noticed that the physics in the two cases is quite different [44].

Linearising equations (1) around an homogeneous equilibrium, i.e. $v^{\pm}(t,z) = \pm a + w^{\pm}(t,z)$ for an electronic plasma yields the simple dispersion relation for a harmonic perturbation $\omega^2 = \omega_p^2 + k^2a^2$. Furthermore computing the thermal velocity

$$v_{th}^2 = \frac{1}{n_0}\int_{-\infty}^{+\infty} v^2\, f_0(v)\,dv = \frac{1}{2a}\int_{-a}^{+a} v^2\,dv = \frac{a^2}{3}$$

allows to recover exactly the Bohm-Gross dispersion relation $\omega^2 = \omega_p^2 + 3k^2v_{th}^2$.

Thus it is very easy to construct the water bag associated to a homogeneous distribution function characterised by a density n_0 and a thermal velocity v_{th}: we just have to choose the water bag parameters (a and $\mathcal{A}$) as follows

$$a = \sqrt{3}v_{th} \quad \text{and} \quad \mathcal{A} = n_0/2a. \tag{5}$$

Of course there is no Landau resonance since the phase velocity $v_\varphi = \sqrt{a^2 + \omega_p^2/k^2} > a$. To recover the Landau damping (particle-wave interaction) the water bag has to be generalised into the multiple water bag.

Let us notice that after a finite time, equations (1) or the system (2)-(3) could generate shocks, namely discontinuous gradients in z for $v^\pm$, n and u. Nevertheless the concept of entropic solution is not well suited here because the existence of an entropy inequality means that a diffusion-like (or collision-like) process in velocity occurs on the right hand side of the Vlasov equation. This observation has been developped in the theory of kinetic formulation of scalar conservation laws [21, 22, 58, 52, 53, 23]. Actually velocity derivatives of non negative bounded measure appear in the right hand side of these linear kinetic equations (free streaming terms), which is the signature of diffusion-like processes in velocity. In order that the water-bag model should be equivalent to the Vlasov equation (without any diffusion-like term on the right hand side of the Vlasov equation) we must consider multivalued solution of the water-bag model beyond the first singularity. The appearence of a singularity (discontinuous gradients in z due to the Burgers term) is linked to appearance of trapped particles which is characterized by the formation of vortices and the development of the filamentation process in phase space. In special cases such as the study of nonlinear gyrokinetic turbulence in a cylinder [15], particles dynamic properties [46] imply that the particles are not trapped but only passing.

2. **The Multi-Water-Bag model.** This generalisation was straightforward [57, 7, 10]. Berk and Roberts [3] and Finzi [32] used a double water bag model to study two stream instability in a computer simulation including the filamentation of the contours and their multivalued nature (a highly difficult problem from a programming point of view).

Let us consider $2\mathcal{N}$ contours in phase space labelled v_j^+ and v_j^- (where $j = 1, \cdots, \mathcal{N}$). Fig. 2 shows the phase space contours for a three-bag system ($\mathcal{N} = 3$) where the distribution function takes on three different constant values F_1, F_2 and F_3.

Introducing the bag heights $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ as shown also in Fig. 2 the distribution function writes

$$f(t, z, v) = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j \left(\mathcal{H}(v_j^+(t, z) - v) - \mathcal{H}(v_j^-(t, z) - v) \right), \tag{6}$$

where $\mathcal{H}$ is the Heaviside unit step function. Notice that some of the $\mathcal{A}_j$ can be negative. The function (6) is a solution of the Vlasov equation in the sense of distribution theory, if and only if the set of following equations are satisfied

$$\partial_t v_j^\pm + v_j^\pm \partial_z v_j^\pm + \frac{q}{m}\partial_z \phi = 0, \quad j = 1, \ldots, \mathcal{N}, \tag{7}$$

where ϕ is the electrical potential with $E = -\partial_z \phi$. Let us now introduce for each bag j the density n_j, average velocity u_j and pressure P_j as done above for the one-bag case $n_j = \mathcal{A}_j(v_j^+ - v_j^-)$, $u_j = (v_j^+ + v_j^-)/2$ and $P_j n_j^{-3} = m/(12\mathcal{A}_j^2)$. For

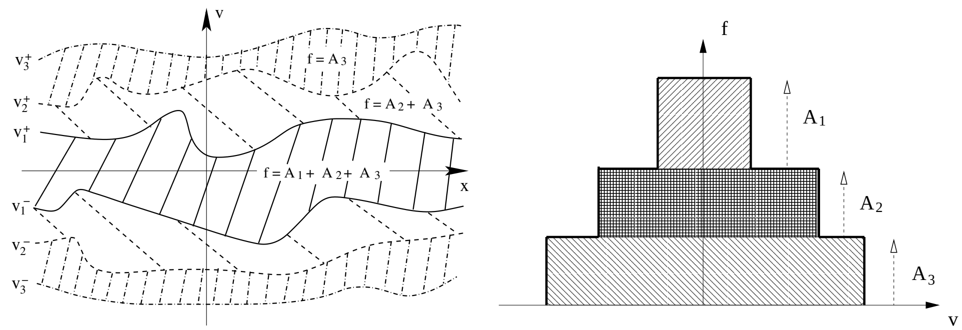


FIGURE 2. *Multiple Water Bag: phase space plot for a three-bag model (left) and corresponding MWB distribution function (right).*

each bag j we recover the conservative form of the continuity and Euler equation (isentropic gas dynamics equations with $\gamma = 3$) namely

$$\partial_t n_j + \partial_z (n_j u_j) = 0, \tag{8}$$

$$\partial_t (n_j u_j) + \partial_z \left(n_j u_j^2 + \frac{P_j}{m} \right) + \frac{q}{m} n_j \partial_z \phi = 0. \tag{9}$$

The coupling between the bags is given by the total density $\sum_{j \leq \mathcal{N}} n_j$ through the Poisson equation (Langmuir or high frequency plasma waves)

$$-\partial_z^2 \phi = \frac{e}{\varepsilon_0} \Big(n_{i0} - \sum_{j=1}^{\mathcal{N}} n_j \Big), \tag{10}$$

with e the elementary charge and n_{i0} a background of fixed ions. For ion acoustic waves in the long wavelength limit, the laplacian can be dropped in the Poisson equation leading to the quasi-neutral balance

$$Z_i n_i = Z_i \sum_{j=1}^{\mathcal{N}} n_j = n_e, \tag{11}$$

with Z_i the number of charge. If we suppose that the electron density n_e follows the Maxwellian-Boltzmann distribution (adiabatic electrons) $n_{e0} \exp(e\phi/(k_B T_e))$, and if we assume the ordering $e\phi/(k_B T_e) \ll 1$, then at first order in $e\phi/(k_B T_e)$ the quasi-neutral equation (11) becomes

$$\phi = \frac{k_B T_e}{n_{e0} e} \Big(Z_i \sum_{j=1}^{\mathcal{N}} n_j - n_{e0} \Big). \tag{12}$$

Linearising equations (8)-(10) for an electronic plasma around an homogeneous (density n_0) equilibrium i.e. $v_j^\pm(t,z) = \pm v_{0j} + \delta v_j^\pm(t,z)$ with $|\delta v_j^\pm| \ll v_{0j}$ yields the dispersion relation

$$\epsilon(k,\omega) = 1 - \frac{\omega_p^2}{n_0} \sum_{j=1}^{\mathcal{N}} \frac{2 v_{0j} \mathcal{A}_j}{\omega^2 - k^2\, v_{0j}^2} = 0. \tag{13}$$

If all $\mathcal{A}_j$'s are positive (single hump distribution function or unimodal function) this equation has $2\mathcal{N}$ real frequencies located between $\pm v_{0j}$ and $\pm v_{0j+1}$. The Landau damping is recovered as a phase mixing process of real frequencies [57, 9] which is reminiscent of the Van Kampen-Case treatment of the electronic plasma oscillations [61, 24].

Let us now introduce the electromagnetic-MWB in the framework of laser-plasma interaction. We aim at describing the behaviour of an electromagnetic wave propagating in a relativistic electron gas in a fixed neutralizing ion background. Here we consider a one-dimensional plasma in space along the z-direction. Since non-linear kinetic effects are important in laser-plasma interaction, we choose a kinetic description for the plasma, which implies to solve a relativistic Vlasov equation for a four-dimensional distribution function $\mathcal{F} = \mathcal{F}(t, z, p_z, p_\perp)$

$$\frac{\partial \mathcal{F}}{\partial t} + \frac{p_z}{m\gamma}\frac{\partial \mathcal{F}}{\partial z} + e\left(E + \frac{p \times B}{m\gamma}\right) \cdot \frac{\partial \mathcal{F}}{\partial p} = 0, \tag{14}$$

where $p = (p_z, p_\perp)$ is the momentum variable, (E, B) the electromagnetic field and γ the Lorentz factor such that $\gamma^2 = 1 + (p_x^2 + p_y^2 + p_z^2)/m^2c^2$. We now reduce the four-dimensional Vlasov equation into a two-dimensional Vlasov equation by using the invariants of the system. The Hamiltonian of a relativistic particle in the electromagnetic field (E, B) for a one-dimensional spatial system reads $\mathscr{H} = mc^2\sqrt{1 + (\mathrm{P}_c - eA)^2/(m^2c^2)} + e\phi(t, z)$ where ϕ is the electrostatic potential, A the vector potential, and P_c the canonical momentum related to the particle momentum p by $\mathrm{P}_c = p + eA$. In order that the field is well determined by the potentials we have to add a gauge. We choose the Coulomb gauge ($\operatorname{div} A = 0$), which implies that $A = A_\perp(t, z)$. Writing the Hamilton equation $\mathrm{dP}_c/dt = -\partial_q \mathscr{H}$, along the longitudinal z-direction of propagation of the electromagnetic wave yields $\mathrm{dP}_{cz}/dt = -\partial_z \mathscr{H}$, while for the transverse direction $\mathrm{dP}_{c\perp}/dt = -\partial_\perp \mathscr{H} = 0$. The last equation means $\mathrm{P}_{c\perp} = \text{constant} = \mathscr{P}_{c\perp}$ and $\mathrm{P}_{c\perp}$ is no more an independent or free variable but a parameter. Therefore the structure of the solution is of the form

$$\mathcal{F}(t, z, p_z, p_\perp) = \int_{\mathscr{P}_{c\perp}} f(t, z, p_z, \mathscr{P}_{c\perp})\delta(p_\perp - (\mathscr{P}_{c\perp} - eA_\perp))\, d\mathscr{P}_{c\perp},$$

where $\mathscr{P}_{c\perp}$ has to be understood as a parameter or a label in f. Therefore, without loss of generality, we now consider a plasma initially prepared so that particles are divided into $\mathcal{M}$ bunches of particles, each bunch i, $1 \le i \le \mathcal{M}$, having the same initial perpendicular canonical momentum $\mathrm{P}_{c\perp} = \mathscr{P}_{c\perp,i}$. The i-particles have any longitudinal momentum p_z with a distribution $f_i(t, z, p_z)$. The Hamiltonian of one particle of bunch i is given by $\mathscr{H}_i(t, z, p_z) = mc^2(\gamma_i(t, z, p_z) - 1) + e\phi(t, z)$ with the corresponding Lorentz factor $\gamma_i^2 = 1 + p_z^2/(m^2c^2) + (\mathscr{P}_{c\perp,i} - eA_\perp(t, z))^2/(m^2c^2)$. Each group i is described by a distribution function $f_i(t, z, p_z)$ which must obey the Vlasov equations $\partial_t f_i + [\mathscr{H}_i, f_i] = 0$, $i = 1, \dots, \mathcal{M}$, where $[\cdot, \cdot]$ is the Poisson bracket in (z, p_z) variables, namely $[\varphi, \psi] = \partial_{p_z}\varphi\partial_z\psi - \partial_z\varphi\partial_{p_z}\psi$. Therefore the structure of the solution is now $\mathcal{F}(t, z, p_z, p_\perp) = \sum_{i=1}^{\mathcal{M}} f_i(t, z, p_z)\delta(p_\perp - (\mathscr{P}_{c\perp,i} - eA_\perp))$. We now assume that each function $f_i(t, z, p_z)$ has the structure of a multi-water-bag

$$f_i(t, z, p_z) = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_{ij}\left(\mathcal{H}(p_{ij}^+(t, z) - p_z) - \mathcal{H}(p_{ij}^-(t, z) - p_z)\right). \tag{15}$$

Pluging equation (15) into the Vlasov equations $\partial_t f_i + [\mathscr{H}_i, f_i] = 0$, $i = 1, \dots, \mathcal{M}$, yield for $i = 1, \dots, \mathcal{M}$ and $j = 1, \dots, \mathcal{N}$, the following multi-water-bag equations

$$\partial_t p_{ij}^\pm + \frac{p_{ij}^\pm}{m\gamma_{ij}^\pm}\partial_z p_{ij}^\pm + \left(eE_z + \frac{1}{2m\gamma_{ij}^\pm}\partial_z\left(\mathscr{P}_{c\perp,i} - eA_\perp(t, z)\right)^2\right) = 0,$$

where $\gamma_{ij}^{\pm} = \sqrt{1 + p_{ij}^{\pm^2}/(m^2c^2) + (\mathscr{P}_{c\perp,i} - eA_\perp(t,z))^2/(m^2c^2)}$. We now add the Maxwell equations which couple the different f_i through the scalar potential ϕ and the vector potential $A_\perp$. The one-dimensional wave-propagation model allows to separate the electric field into two parts, namely $E = E_z\mathbf{e}_z + E_\perp$, where $E_z = -\partial_z\phi$ is a pure electrostatic field, which obeys Poisson's equation, and $E_\perp = -\partial_t A_\perp$ is a pure electromagnetic field. In absence of any external magnetic field, B is purely perpendicular and is given by $B_\perp = \nabla \times A_\perp$. The two others couple the distribution functions f_i. The Maxwell-Gauss equation $\partial_z E_z = \rho/\varepsilon_0$ becomes

$$\partial_z E_z = \frac{e}{\varepsilon_0}\left(\sum_{i=1}^{\mathcal{M}} n_i(t,z) - n_0\right),$$

where the charge density n_i of the bunch i is defined by

$$n_i(t,z) = \int_{-\infty}^{\infty} f_i(t,z,p_z)\,dp_z = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_{ij}(p_{ij}^{+}(t,z) - p_{ij}^{-}(t,z)).$$

The two Maxwell equations $\nabla \times E_\perp + \partial_t B_\perp = 0$ and $\nabla \times B_\perp = \mu_0(J_\perp + \varepsilon_0\partial_t E_\perp)$ can be combined to get the waves equation

$$\partial_t^2 A_\perp - c^2\partial_z^2 A_\perp = \mu_0 \sum_{i=1}^{\mathcal{M}} J_{\perp,i},$$

where $J_{\perp,i}$ is defined by

$$\begin{aligned} J_{\perp,i}(t,z) &= \frac{e}{m}(\mathscr{P}_{c\perp,i} - eA_\perp))\int_{-\infty}^{\infty} f_i(t,z,p_z)\,\frac{dp_z}{\gamma_i} \\ &= \frac{e}{m}(\mathscr{P}_{c\perp,i} - eA_\perp))\sum_{j=1}^{\mathcal{N}} \mathcal{A}_{ij}\int_{p_{ij}^{-}(t,z)}^{p_{ij}^{+}(t,z)} \frac{dp_z}{\gamma_i}. \end{aligned}$$

In the sequel we will consider the particular case $\mathcal{M} = 1$, thus corresponding to a cold plasma distribution in the perpendicular direction. Since usually no streaming effects are considered we take $\mathscr{P}_{c\perp,1} = 0$. Moreover we suppose that the relativistic effects are negligible, thus $\gamma_i^{\pm} = 1$ and $\gamma_{ij}^{\pm} = 1$. Using the relations $\omega_p^2 = e^2 n_0/(m\varepsilon_0)$ and $\mu_0\varepsilon_0 c^2 = 1$, our electromagnetic-MWB model is described by the following equations

$$\partial_t v_j^{\pm} + v_j^{\pm}\partial_z v_j^{\pm} + \frac{e}{m}\partial_z\left(\phi + \frac{e}{2m}|A_\perp|^2\right) = 0, \tag{16}$$

$$-\partial_z^2\phi = \frac{en_0}{\varepsilon_0}(\rho_v - 1), \quad \partial_t^2 A_\perp - c^2\partial_z^2 A_\perp = \omega_p^2 A_\perp \rho_v, \tag{17}$$

$$\rho_v = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j(v_j^{+} - v_j^{-}). \tag{18}$$

3. **The Cauchy problem.** We now present existence and uniqueness proofs of classical solutions for the multi-water-bag models depicted in the previous section, namely the Poisson-multi-water-bag, the quasineutral-multi-water-bag and the electromagnetic-multi-water-bag models.

3.1. **Definitions and Notations.** Let Ω be an open set of $\mathbb{R}^d$. $L^\infty(\Omega)$ is the space of real functions on Ω which are measurable and essentially bounded. It is a Banach space for the norm $\|f\|_{L^\infty(\Omega)} = \sup\operatorname{ess}_{x\in\Omega} |f(x)|$. $L^2(\Omega)$ is a Hilbert space for the scalar product $\langle f,g\rangle = \int_\Omega f(x)g(x)dx$ and the corresponding norm is $\|f\|_{L^2(\Omega)} = \langle f,f\rangle^{1/2}$. If we use the notation $\partial^\alpha f = \partial^{\alpha_1}_{x_1}\ldots\partial^{\alpha_d}_{x_d} f$ where $\alpha = (\alpha_1,\ldots,\alpha_d)\in\mathbb{N}^d$ and $|\alpha| = \alpha_1+\ldots+\alpha_d$, then for $m\in\mathbb{N}$, $W^{m,\infty}(\Omega)$ is the space of functions which belong to $L^\infty(\Omega)$ and whose distributional derivatives of order less or equal to m belong to $L^\infty(\Omega)$. This a Banach space for the norm $\|f\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\le m}\|\partial^\alpha f\|_{L^\infty(\Omega)}$. For $m\in\mathbb{N}$, $H^m(\Omega)$ is the space of functions which belong to $L^2(\Omega)$ and whose distributional derivatives of order less or equal to m belong to $L^2(\Omega)$. This is a Hilbert space for the scalar product $\langle f,g\rangle_{H^m(\Omega)} = \sum_{|\alpha|\le m}\langle\partial^\alpha f,\partial^\alpha g\rangle$ and the corresponding norm is $\|f\|_{H^m(\Omega)} = \langle f,f\rangle^{1/2}_{H^m(\Omega)}$. If X is a Banach or Hilbert space, $L^\infty(a,b;X)$ (resp. $\mathscr{C}(a,b;X)$) is the space of measurable (resp. continuous) functions from (a,b) into X which are essentially bounded (resp. bounded). This is a Banach space for the norm $\|f\|_{L^\infty(a,b;X)} = \sup\operatorname{ess}_{t\in(a,b)}\|f(t)\|_X$ (resp. $\|f\|_{\mathscr{C}(a,b;X)} = \sup_{t\in(a,b)}\|f(t)\|_X$). Finally $\operatorname{Lip}(a,b;X)$ is the space of measurable functions from (a,b) into X which are essentially bounded and Lipschitz. This is a Banach space for the norm $\|f\|_{\operatorname{Lip}(a,b;X)} = \|f\|_{L^\infty(a,b;X)} + \sup\limits_{\substack{s,t\in(a,b)\\ s\neq t}} |\|f(s)\|_X - \|f(t)\|_X|\,|s-t|^{-1}$.

3.2. **The Poisson-MWB model.**

3.2.1. *The case of a finite number of bag.* In this section, we consider the initial value periodic problem

$$
\begin{aligned}
&\partial_t v_j^\pm + v_j^\pm\partial_z v_j^\pm + \partial_z\phi = 0, \quad v_j^\pm(0,\cdot) = v_{0j}^\pm(\cdot), \quad j=1,\ldots,\mathcal{N},\\
&-\partial_z^2\phi = \sum_{j=1}^{\mathcal{N}}\mathcal{A}_j(v_j^+ - v_j^-) - 1,
\end{aligned}
\tag{19}
$$

with period $\Omega = 1$, $z\in\mathbb{R}/\mathbb{Z}$. Therefore we have the following existence theorem.

Theorem 3.1. *(Local classical solution). Assume $v_{0j}^\pm\in H^m(\Omega)$ with $m>3/2$ and $1\le j\le\mathcal{N}$. Then for all $\mathcal{N}$ there exists a time $T>0$ which depends only on $\|v_{0j}^\pm\|_{H^m(\Omega)}$, $\mathcal{N}$, Ω and $A=\max_{j\le\mathcal{N}}|\mathcal{A}_j|$, such that the system (19) admits a unique solution*

$$
\begin{aligned}
&v_j^\pm\in L^\infty(0,T;H^m(\Omega))\cap\operatorname{Lip}(0,T;H^{m-1}(\Omega)), \quad j=1,\ldots,\mathcal{N},\\
&\phi\in L^\infty(0,T;H^{m+2}(\Omega))\cap\operatorname{Lip}(0,T;H^{m+1}(\Omega)).
\end{aligned}
$$

Proof. The proof is based on a fixed point argument (Banach's fixed-point theorem). We first rewrite the system (19). Using the Green function $G(z,y)$, i.e. the fundamental solution of the differential operator $-\partial_z^2$ with periodic boundary conditions ($-\partial_z^2 G(z,y) = \delta(z-y)$, $G(0,y)=G(1,y)$), we can reconsider the problem (19) along

$$\partial_t v_j^\pm + v_j^\pm\partial_z v_j^\pm = \partial_z\phi\,[\rho_v]\,,$$

where

$$\rho_v = \sum_{j=1}^{\mathcal{N}}\mathcal{A}_j\left(v_j^+(t,y) - v_j^-(t,y)\right) - 1, \tag{20}$$

and

$$\phi\left[\rho_v\right](t,z) = \int_\Omega G(z,y)\rho_v(t,y)dy, \tag{21}$$

with $G(z,y) = z(1-y)$ for $0 \le z < y$ and $G(z,y) = y(1-z)$ for $y \le z \le 1$. The regularity properties of the solution of the Poisson equation in L^2 imply that

$$\|\phi[\rho]\|_{H^m(\Omega)} \le C(\Omega)\,\|\rho\|_{H^{\max\{m-2,0\}}(\Omega)}\,. \tag{22}$$

We now define the set W_T as

$$\begin{aligned} W_T := \Big\{ w_j^\pm \in L^\infty(0,T;H^m(\Omega)) \cap \mathrm{Lip}(0,T;H^{m-1}(\Omega)), \quad j = 1,\dots,\mathcal{N} \mid \\ \sup_{t\in[0,T]} \|\{w_j^\pm(t,\cdot)\}_{j\le\mathcal{N}}\|_{\mathbb{H}^m} := \sup_{t\in[0,T]} \sum_{j=1}^{\mathcal{N}} \|w_j^+(t,\cdot)\|_{H^m(\Omega)} + \|w_j^-(t,\cdot)\|_{H^m(\Omega)} \\ \le K\|\{v_{0j}^\pm\}_{j\le\mathcal{N}}\|_{\mathbb{H}^m} \Big\} \end{aligned}$$

with $K > 1$ a numerical constant. We then define the iteration map $\mathcal{F}$ as follows. For any sequence $\{w_j^\pm\}_{j\le\mathcal{N}} \in W_T$ the image $\mathcal{F}(\{w_j^\pm\}_{j\le\mathcal{N}})$ is the unique solution $\{v_j^\pm\}_{j\le\mathcal{N}}$ of

$$\partial_t v_j^\pm + v_j^\pm \partial_z v_j^\pm = \partial_z \phi\left[\rho_w\right], \tag{23}$$

with $v_{0j}^\pm$ as initial condition. We first show that $\mathcal{F}$ maps W_T onto itself for T small enough. If we apply ∂_z^α to (23) for $\alpha \le m$ and take the L^2-scalar product with $\partial_z^\alpha v_j^\pm$ then we get

$$\frac{1}{2}\frac{d}{dt}\|\partial_z^\alpha v_j^\pm\|^2_{L^2(\Omega)} + \int_\Omega \partial_z^\alpha(v_j^\pm\partial_z v_j^\pm)\partial_z^\alpha v_j^\pm dz = \int_\Omega \partial_z^{\alpha+1}\phi\left[\rho_w\right]\partial_z^\alpha v_j^\pm dz. \tag{24}$$

Let us estimate first the right hand side of (24). Applying Cauchy-Schwarz inequality and using (22) we get

$$\begin{aligned} \left|\int_\Omega \partial_z^{\alpha+1}\phi\left[\rho_w\right]\partial_z^\alpha v_j^\pm dz\right| &\le \|\partial_z^\alpha v_j^\pm\|_{L^2(\Omega)}\|\partial_z^{\alpha+1}\phi[\rho_w]\|_{L^2(\Omega)} \\ &\le \|\partial_z^\alpha v_j^\pm\|_{L^2(\Omega)}\|\rho_w\|_{H^{\alpha-1}(\Omega)} \\ &\le C(\Omega,A)\|v_j^\pm\|_{H^m(\Omega)}\|\{w_j^\pm\}_{j\le\mathcal{N}}\|_{\mathbb{H}^m}. \end{aligned} \tag{25}$$

We now estimate the second term of the left hand side of (24). Using Leibniz rules to evaluate $\partial_z^\alpha(v_j^\pm\partial_z v_j^\pm)$ we have

$$\partial_z^\alpha(v_j^\pm\partial_z v_j^\pm) = v_j^\pm\partial_z^{\alpha+1}v_j^\pm + \sum_{k=1}^{\alpha}\begin{pmatrix}\alpha\\ k\end{pmatrix}\partial_z^k v_j^\pm\partial_z^{\alpha-k+1}v_j^\pm. \tag{26}$$

Pluging (26) into (24) the first part can be estimated as

$$\left|\int_\Omega v_j^\pm\partial_z^{\alpha+1}v_j^\pm\partial_z^\alpha v_j^\pm dz\right| = \frac{1}{2}\left|\int_\Omega v_j^\pm\partial_z(\partial_z^\alpha v_j^\pm)^2 dz\right| \le \frac{1}{2}\|v_j^\pm\|_{W^{1,\infty}(\Omega)}\|v_j^\pm\|^2_{H^m(\Omega)}.$$

Using the Cauchy-Schwarz inequality and the interpolation inequality (see Proposition 3.6, Chapter 13 of [60])

$$\begin{aligned} \|\partial_z^{k-1}\partial_z f\partial_z^{\alpha-k}\partial_z g\|_{L^2(\Omega)} \le C(m)(\|\partial_z f\|_{L^\infty(\Omega)}\|g\|_{H^\alpha(\Omega)} \\ + \|f\|_{H^\alpha(\Omega)}\|\partial_z g\|_{L^\infty(\Omega)}), \end{aligned} \tag{27}$$

we obtain

$$\begin{aligned}\left|\int_\Omega \partial_z^\alpha v_j^\pm \sum_{k=1}^{\alpha}\left(\begin{array}{c}\alpha\\ k\end{array}\right)\partial_z^{k-1}\partial_z v_j^\pm \partial_z^{\alpha-k}\partial_z v_j^\pm dz\right| &\leq C(m)\|v_j^\pm\|^2_{H^m(\Omega)}\|\partial_z v_j^\pm\|_{L^\infty(\Omega)},\\ &\leq C(m)\|v_j^\pm\|^3_{H^m(\Omega)}, \qquad (28)\end{aligned}$$

where we have used the Sobolev imbedding $H^m(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$ for $m>3/2$. Using (24)-(25) and (28) we get

$$\frac{d}{dt}\|v_j^\pm\|_{H^m(\Omega)} \leq C(m,\Omega,A)\left(\|v_j^\pm\|^2_{H^m(\Omega)} + \|\{w_j^\pm\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m}\right),\ j=1,\dots,\mathcal{N}. \quad (29)$$

Summing the previous inequality over j we finally obtain the differential inequality

$$\frac{d}{dt}\|\{v_j^\pm(t)\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m} \leq C(m,\Omega,A,\mathcal{N})\left(\|\{v_j^\pm(t)\}_{j\leq\mathcal{N}}\|^2_{\mathbb{H}^m} + \|\{w_j^\pm(t)\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m}\right).$$

Then a Gronwall lemma (see appendix A) shows that $\|\{v_j^\pm(t)\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m} \leq K\|\{v_{0j}^\pm\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m}$ for all $t\in[0,T]$, T small enough. From (23) we have $v_j^\pm\in \mathrm{Lip}(0,T;H^{m-1}(\Omega))$ for $1\leq j\leq\mathcal{N}$. We then conclude that the application $\mathcal{F}$ maps W_T into itself. We now need to prove that $\mathcal{F}$ is a contraction. We consider two set of functions $\{w_{1j}^\pm\}_{j\leq\mathcal{N}}$ and $\{w_{2j}^\pm\}_{j\leq\mathcal{N}}$ belonging to W_T. We set $\{v_{1j}^\pm\}_{j\leq\mathcal{N}} := \mathcal{F}(\{w_{1j}^\pm\}_{j\leq\mathcal{N}})$, $\{v_{2j}^\pm\}_{j\leq\mathcal{N}} := \mathcal{F}(\{w_{2j}^\pm\}_{j\leq\mathcal{N}})$, $v_j^\pm = v_{1j}^\pm - v_{2j}^\pm$ and $w_j^\pm = w_{1j}^\pm - w_{2j}^\pm$ for all $1\leq j\leq\mathcal{N}$. The difference of the equations (23) for $\{v_{1j}^\pm\}$ and $\{v_{2j}^\pm\}$ gives

$$\partial_t v_j^\pm + v_j^\pm\partial_z v_{1j}^\pm + v_{2j}^\pm\partial_z v_j^\pm = \partial_z\phi\left[\rho_w\right], \quad v_j^\pm(0,\cdot)=0. \quad (30)$$

In the same manner we obtained (24) we deduce from (30)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_z^\alpha v_j\|_{L^2(\Omega)} &+ \int_\Omega \partial_z^\alpha(v_j^\pm\partial_z v_{1j}^\pm)\partial_z^\alpha v_j^\pm dz\\ &+ \int_\Omega \partial_z^\alpha(v_{2j}^\pm\partial_z v_j^\pm)\partial_z^\alpha v_j^\pm dz = \int_\Omega \partial_z^{\alpha+1}\phi\left[\rho_w\right]\partial_z^\alpha v_j^\pm dz. \quad (31)\end{aligned}$$

Using the estimates of Proposition 3.7, Chapter 3 of [60] the second term of the left hand side of (31) for $\alpha\leq m-1$ is bounded as follows

$$\begin{aligned}\left|\int_\Omega \partial_z^\alpha(v_j^\pm\partial_z v_{1j}^\pm)\partial_z^\alpha v_j^\pm dz\right| &\leq C(m)\|v_j^\pm\|_{H^{m-1}(\Omega)}\|v_j^\pm\partial_z v_{1j}^\pm\|_{H^{m-1}(\Omega)}\\ &\leq C(m)\|v_j^\pm\|_{H^{m-1}(\Omega)}(\|\partial_z v_{1j}^\pm\|_{L^\infty(\Omega)}\|v_j^\pm\|_{H^{m-1}(\Omega)}\\ &\quad +\|v_j^\pm\|_{L^\infty(\Omega)}\|\partial_z v_{1j}^\pm\|_{H^{m-1}(\Omega)})\\ &\leq C(m)\|v_j^\pm\|^2_{H^{m-1}(\Omega)}\|v_{1j}^\pm\|_{H^m(\Omega)}.\end{aligned}$$

For the second term of the left hand side of (31) we proceed similarly to (28) and provided $m>3/2$ we get

$$\begin{aligned}\left|\int_\Omega \partial_z^\alpha(v_{2j}^\pm\partial_z v_j^\pm)\partial_z^\alpha v_j^\pm dz\right| &\leq \|v_{2j}^\pm\|_{W^{1,\infty}(\Omega)}\|v_j^\pm\|^2_{H^{m-1}(\Omega)} + \|v_j^\pm\|_{H^{m-1}(\Omega)}\\ &\qquad \left\|\sum_{k=1}^{\alpha}\left(\begin{array}{c}\alpha\\ k\end{array}\right)\partial_z^{k-1}(\partial_z v_{2j}^\pm)\partial_z^{\alpha-k+1}v_j^\pm\right\|_{L^2(\Omega)}\\ &\leq C(m)\|v_j^\pm\|^2_{H^{m-1}(\Omega)}\|v_{2j}^\pm\|_{H^m(\Omega)}.\end{aligned}$$

The estimate of right hand side of (31) is the same as (25).
Since $\|\{v_{1j}^{\pm}(t)\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m}$, $\|\{v_{2j}^{\pm}(t)\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m} \leq K\|\{v_{0j}^{\pm}\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m}$ we finally obtain

$$\begin{aligned}&\frac{d}{dt}\|\{v_j^{\pm}(t)\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^{m-1}}\\ &\qquad\qquad \leq C(m,\Omega,A,\mathcal{N},\mathcal{K}_0)\left(\|\{v_j^{\pm}(t)\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^{m-1}}+\|\{w_j^{\pm}(t)\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^{m-1}}\right).\end{aligned}$$

where $\mathcal{K}_0 = K\|\{v_{0j}^{\pm}\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m}$. Once again, a Gronwall lemma shows that $\mathcal{F}$ is a contraction provided T is small enough. □

3.2.2. *The case of an infinite countable number of bag.* The theorem 3.1 is not true for an infinite number of bag because the constants involving in the proof depend on the number of bag. Nevertheless the theorem 3.1 can be extended to the case of an infinite countable number of bag by replacing the norm $\|\cdot\|_{\mathbb{H}^m}$ with the norm $\|\cdot\|_{\mathrm{L}^{1\cap\infty}\mathrm{H}^m}$ defined by

$$\begin{aligned}\|\{v_j^{\pm}(t)\}\|_{\mathrm{L}^{1\cap\infty}\mathrm{H}^m} = &\sup_{\alpha\in\{+,-\}}\sup_{j\in\mathbb{N}^*}\|v_j^{\alpha}(t)\|_{H^m(\Omega)}\\ &+\sum_{j\in\mathbb{N}^*}|\mathcal{A}_j|(\|v_j^{+}(t)\|_{H^m(\Omega)}+\|v_j^{-}(t)\|_{H^m(\Omega)}),\end{aligned} \tag{32}$$

where we have assumed that the sum $\sum_{j\in\mathbb{N}^*}|\mathcal{A}_j|$ is bounded. Therefore we have the following existence theorem.

Theorem 3.2. *(Local classical solution). Assume $v_{0j}^{\pm}\in H^m(\Omega)$ with $m>3/2$ and $j\in\mathbb{N}^*$. Let us suppose that the coefficients $\{\mathcal{A}_j\}_{j\in\mathbb{N}^*}$ are such that the sum $\sum_{j\in\mathbb{N}^*}|\mathcal{A}_j| = \mathcal{A}$ is bounded. Then there exists a time $T>0$ which depends only on $\|v_{0j}^{\pm}\|_{H^m(\Omega)}$, $\mathcal{A}$ and Ω such that the system (19) admits a unique solution*

$$\begin{aligned}&v_j^{\pm}\in L^{\infty}(0,T;H^m(\Omega))\cap\mathrm{Lip}(0,T;H^{m-1}(\Omega)),\quad j\in\mathbb{N}^*,\\ &\phi\in L^{\infty}(0,T;H^{m+2}(\Omega))\cap\mathrm{Lip}(0,T;H^{m+1}(\Omega)).\end{aligned}$$

Proof. Following the method which leads to inequality (29) we obtain

$$\frac{d}{dt}\|v_j^{\pm}\|_{H^m(\Omega)}\leq C(m,\Omega)\left(\|v_j^{\pm}\|^2_{H^m(\Omega)}+\|\{w_j^{\pm}\}\|_{\mathrm{L}^{1\cap\infty}\mathrm{H}^m}\right),\quad j\in\mathbb{N}^*. \tag{33}$$

Integrating (33) in time and using the definition of the norm (32) we get

$$\begin{aligned}\|\{v_j^{\pm}(t)\}\|_{\mathrm{L}^{1\cap\infty}\mathrm{H}^m}&\leq\|\{v_j^{\pm}(0)\}\|_{\mathrm{L}^{1\cap\infty}\mathrm{H}^m}\\ &+C(m,\Omega,\mathcal{A})\int_0^t(\|\{v_j^{\pm}(s)\}\|^2_{\mathrm{L}^{1\cap\infty}\mathrm{H}^m}+\|\{w_j^{\pm}(s)\}\|_{\mathrm{L}^{1\cap\infty}\mathrm{H}^m})ds.\end{aligned} \tag{34}$$

Following the proof of theorem 3.1 and using the estimate (34), a Gronwall lemma (see appendix A) allows to construct a set W_T and an application $\mathcal{F}$ which maps W_T into itself. Following the end of the proof of theorem 3.1 we can show that the application $\mathcal{F}$ is also a contraction and according to the Banach's fixed-point theorem we deduce the local existence and uniqueness result. □

3.2.3. *The case of a continuum of bag.* In order to consider a continuum of bag we define two Lagrangian foliations to be the families of sheets $v^{\pm}=v^{\pm}(t,z,a)$ labelled by the Lagrangian label $a\in[0,1]$ where the water-bag continuum $v^{\pm}(t,z,a)$ are smooth functions. The system (19) is still valid if we replace the counting measure by the Lebesgue measure da. In fact let us consider the function

$$f(t,z,v)=\int_0^1 \Big(\mathcal{H}(v^+(t,z,a)-v)-\mathcal{H}(v^-(t,z,a)-v)\Big)\,d\mu(a), \tag{35}$$

where

$$\mu(a)=\begin{cases}\mu^{\mathcal{N}}(a) &= \sum_{j=1}^{\mathcal{N}}\mathcal{A}_j\delta(a-a_j)\\ \text{or} & \\ \mu^{\infty}(a) &= \mathbb{1}_{[0,1]}(a).\end{cases}$$

Obviously we have $\mu^{\mathcal{N}}\rightharpoonup\mu^{\infty}$ for the weak-$*$ topology $\sigma(\mathcal{M}_b,\mathscr{C}_b)$ (topology of the narrow convergence) where $\mathcal{M}_b$ is the set of bounded Radon measure. Therefore it is easily verified by a direct check that f defined by equation (35) satisfies the Vlasov in the distributional sense equation

$$\partial_t f+v\partial_z f-\partial_z\phi\partial_v f=0,\quad -\partial_z^2\phi=\int_{\mathbb{R}} f\,dv, \tag{36}$$

if and only if the water-bag continuum $v^\pm$ satisfy the continuous water-bag model given by

$$\partial_t v^\pm+v^\pm\partial_z v^\pm+\partial_z\phi=0,\quad -\partial_z^2\phi=\int_0^1(v^+-v^-)da, \tag{37}$$

with period $\Omega=1$, $z\in\mathbb{R}/\mathbb{Z}$, $a\in[0,1]$. Therefore we have the following existence theorem.

Theorem 3.3. *(Local classical solution). Assume $v_0^\pm\in H^m(\mathcal{D})$ with $m>2$ and $\mathcal{D}=\Omega\times[0,1]$. Then there exists a time $T>0$ which depends only on $\|v_0^\pm\|_{H^m(\mathcal{D})}$, $\mathcal{D}$, such that the system (37) admits a unique solution*

$$v^\pm\in L^\infty(0,T;H^m(\mathcal{D}))\cap\mathrm{Lip}(0,T;H^{m-1}(\mathcal{D})),$$
$$\phi\in L^\infty(0,T;H^{m+2}(\Omega))\cap\mathrm{Lip}(0,T;H^{m+1}(\Omega)).$$

Proof. The proof of the theorem is the same as the theorem 3.1, except that the problem is now set in a two-dimensional space, which implies more regularity for the initial conditions because of the Sobolev embeddings. □

3.3. **The Quasineutral-MWB model.**

3.3.1. *The case of a finite number of bag.* In this section, we consider the initial value periodic problem

$$\begin{aligned}&\partial_t v_j^\pm+v_j^\pm\partial_z v_j^\pm+\partial_z\phi=0,\quad v_j^\pm(0,\cdot)=v_{0j}^\pm(\cdot),\quad j=1,\dots,\mathcal{N},\\ &\phi=\sum_{j=1}^{\mathcal{N}}\mathcal{A}_j(v_j^+-v_j^-)-1,\end{aligned} \tag{38}$$

with period $\Omega=1$, $z\in\mathbb{R}/\mathbb{Z}$. Therefore we have the existence theorem

Theorem 3.4. *(Local classical solution). Assume $v_{0j}^\pm\in H^m(\Omega)$ with $m>3/2$ and $\mathcal{A}_j$ positive real numbers, $1\le j\le\mathcal{N}$. Moreover we suppose that $\sum_{j\le\mathcal{N}}\mathcal{A}_j=A$ is bounded. Then for all $\mathcal{N}$ there exists a time $T>0$ which depends only on $\|v_{0j}^\pm\|_{H^m(\Omega)}$, $\mathcal{N}$, Ω and A, such that the system (38) admits a unique solution*

$$v_j^\pm,\ \phi\in L^\infty(0,T;H^m(\Omega))\cap\mathscr{C}(0,T;H^m(\Omega)),\quad j=1,\dots,\mathcal{N}.$$

Proof. If we set $V = (v_1^+, \ldots, v_{\mathcal{N}}^+, v_1^-, \ldots, v_{\mathcal{N}}^-)^T$ the system of equations (38) can be recast in the quasilinear hyberbolic system

$$\partial_t V + \mathcal{B}(V)\partial_z V = 0, \tag{39}$$

where $\mathcal{B} = \mathcal{D} + \mathbb{1}\mathcal{A}^T$ with

$$\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_{\mathcal{N}}, -\mathcal{A}_1, \ldots, -\mathcal{A}_{\mathcal{N}})^T, \quad \mathbb{1} = (\underbrace{1, \ldots, 1}_{2\mathcal{N} times})^T, \quad \text{and}$$

$$\mathcal{D} = \begin{pmatrix} v_1^+ & & & & & \\ & \ddots & & & 0 & \\ & & v_{\mathcal{N}}^+ & & & \\ & & & v_1^- & & \\ & 0 & & & \ddots & \\ & & & & & v_{\mathcal{N}}^- \end{pmatrix}.$$

Let us show that the system (39) is strictly hyperbolic. To this purpose, we just need to show that the matrix $\mathcal{B}$ has $2\mathcal{N}$ distinct real eigenvalues. Let be λ a real number. Then after some rearrangement of the line of $\mathcal{B} - \lambda\mathcal{I}$, the latter matrix takes the form

$$\mathcal{B} - \lambda\mathcal{I} =$$
$$\begin{pmatrix} v_1^+ - \lambda & -v_2^+ + \lambda & & & & & \\ \ddots & \ddots & \ddots & & 0 & & \\ & v_{\mathcal{N}}^+ - \lambda & -v_1^- + \lambda & & & & \\ & & v_1^- - \lambda & -v_2^- + \lambda & & & \\ & 0 & & \ddots & \ddots & \ddots & \\ & & & & & v_{\mathcal{N}-1}^- - \lambda & -v_{\mathcal{N}}^- + \lambda \\ \mathcal{A}_1 & \cdots & \mathcal{A}_{\mathcal{N}} & -\mathcal{A}_1 & \cdots & -\mathcal{A}_{\mathcal{N}-1} & v_{\mathcal{N}}^- - \lambda - \mathcal{A}_{\mathcal{N}} \end{pmatrix}. \tag{40}$$

The determinant of (40) yields a polynomial of degree $2\mathcal{N}$

$$P_{2\mathcal{N}}(\lambda) = \prod_{j=1}^{\mathcal{N}} (v_j^+ - \lambda)(v_j^- - \lambda) \left(1 - \sum_{j=1}^{\mathcal{N}} \frac{n_j}{(v_j^+ - \lambda)(v_j^- - \lambda)}\right), \tag{41}$$

where $n_i = \mathcal{A}_j(v_j^+ - v_j^-)$, $\mathcal{A}_j \geq 0$, $v_j^+ > 0$ and $v_j^- < 0$. We then observe that

$$\text{sign}\left(P_{2\mathcal{N}}(v_1^\pm)\right) = \text{sign}\left(P_{2\mathcal{N}}(0)\right)$$

and for $j = 2, \ldots, \mathcal{N}$,

$$\text{sign}\left(P_{2\mathcal{N}}(v_j^\pm)\right) = \begin{cases} (-1)^j & \mathcal{N} \text{ odd}, \\ (-1)^{j+1} & \mathcal{N} \text{ even}. \end{cases}$$

Consequently the polynomial $P_{2\mathcal{N}}$ oscillates $2\mathcal{N} - 2$ times around zero and has $2\mathcal{N} - 2$ roots, $\mathcal{N} - 1$ positive $\{\lambda_j^+\}_{j \leq \mathcal{N}-1}$, and $\mathcal{N} - 1$ negative $\{\lambda_j^-\}_{j \leq \mathcal{N}-1}$ such that $v_{j-1}^\pm < \lambda_j^\pm < v_j^\pm$, $2 \leq j \leq \mathcal{N}$. Therefore $P_{2\mathcal{N}}$ can be factorizes as follows

$$P_{2\mathcal{N}}(\lambda) = Q_{2\mathcal{N}-2}(\lambda)S_2(\lambda),$$

with $Q_{2\mathcal{N}-2}(\lambda) = \prod_{j=1}^{\mathcal{N}-1}(\lambda - \lambda_j^+)(\lambda - \lambda_j^-)$ and $S_2(\lambda) = \lambda^2 + a\lambda + b$. If $\mathcal{N}$ is even then $P_{2\mathcal{N}}(0) > 0$ and $Q_{2\mathcal{N}-2}(0) < 0$. Therefore $S_2(0) < 0$ and $S_2(\lambda)$ has two distinct real roots of opposite sign. If $\mathcal{N}$ is now odd then $P_{2\mathcal{N}}(0) < 0$ and $Q_{2\mathcal{N}-2}(0) > 0$. Therefore $S_2(0) < 0$ and $S_2(\lambda)$ has again two distinct real roots of opposite sign. Finally we conclude that $P_{2\mathcal{N}}(\lambda)$ has $2\mathcal{N}$ distinct real roots, $\mathcal{N}$ positive and $\mathcal{N}$ negative. Therefore the system (39) is strictly hyperbolic, and from Proposition 2.2, Chapter16 of [60] it is symmetrizable. We finally deduce the existence and the regularity of the local classical solution from Proposition 2.1 of Chapter 16 of [60]. □

Remark 1. In fact the distribution function (6) can solve more general kinetic equation. Indeed, let us choose the following distribution function

$$f(t,z,v) = \sum_{i=0}^{N-1} c_i \mathbb{1}_{v_i(t,z)<v<v_{i+1}(t,z)}(v), \tag{42}$$

where the function $\mathbb{1}_{a<v<b}(v)$ is equal to one if $v \in]a,b[$ and null elsewhere. If $N = 2\mathcal{N}$ and if there are $\mathcal{N}$ positive bags $\{v_i\}_{i\in\Sigma^+}$ (Σ^+ the index set of positive bags) and $\mathcal{N}$ negative bags $\{v_i\}_{i\in\Sigma^-}$ (Σ^- the index set of negative bags) then the distribution function (6) is equivalent to (42) with $(-1)^l\mathcal{A}_i = c_{i+1} - c_i$, where $l = 1$ if $i \in \Sigma^+$ and $l = 2$ if $i \in \Sigma^-$. Therefore the distribution fonction (42) is a water-bag-like weak solution of the kinetic equation

$$\partial_t f + v\partial_z f - \partial_z q(\rho)\ \partial_v f = 0, \tag{43}$$

where

$$\rho(t,z) = \int_{\mathbb{R}} f(t,z,v)\,dv,$$

if and only if

$$\partial_t v_i + \partial_z\left(\frac{v_i^2}{2} + q(\rho)\right) = 0, \quad i = 0,\cdots,N. \tag{44}$$

Particularly we recover the quasineutral-MWB model if $q(\rho) = \rho$, for which we get

$$q(\rho) = \sum_{i=0}^{N-1} c_i(v_{i+1} - v_i).$$

The existence of classical solution of (44) still relies on the hyperbolicity of the system (44). If we assume that at (t,z) fixed, the application $v \to f(t,z,v)$ has a single change of monotonicity, i.e. there exists n_0 such that $c_{i+1} > c_i$ for $i = 0,\cdots,n_0-1$ and $c_{i+1} < c_i$ for $i = n_0,\cdots,N-2$, then the system is hyperbolic. Indeed the characteristic polynomial of the jacobian is

$$R(\lambda) = \begin{vmatrix} v_0 + \frac{\partial q}{\partial v_0} - \lambda & \frac{\partial q}{\partial v_1} & \cdots & \frac{\partial q}{\partial v_{N-1}} & \frac{\partial q}{\partial v_N} \\ \frac{\partial q}{\partial v_0} & v_1 + \frac{\partial q}{\partial v_1} - \lambda & \cdots & \frac{\partial q}{\partial v_{N-1}} & \frac{\partial q}{\partial v_N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial q}{\partial v_0} & \frac{\partial q}{\partial v_1} & \cdots & v_{N-1} + \frac{\partial q}{\partial v_{N-1}} - \lambda & \frac{\partial q}{\partial v_N} \\ \frac{\partial q}{\partial v_0} & \frac{\partial q}{\partial v_1} & \cdots & \frac{\partial q}{\partial v_{N-1}} & v_N + \frac{\partial q}{\partial v_N} - \lambda \end{vmatrix}$$

and we have

$$R(v_i) = \frac{\partial q}{\partial v_i} \prod_{j=0, j\neq i}^{N} (v_j - v_i).$$

If we now assume that $p' > 0$, since $p'(\rho) = \rho q'(\rho)$, $\frac{\partial q}{\partial v_i}$ has the same sign than $\frac{\partial \rho}{\partial v_i} = c_{i-1} - c_i$ and then the sign of $R(v_i)$ is $(-1)^{i-1}\mathrm{sign}(c_{i-1} - c_i)$. Besides the dominant term of $P(\lambda)$ is $(-1)^{N+1}\lambda^{N+1}$ and thus if

$$v_0 < v_1 < \cdots < v_N,$$

then we have $N+1$ distinct roots for R. Therefore the system (44) is hyperbolic, and thus symmetrizable, and finally it has a unique local classical solution v_i, $i = 1, \ldots, N$, such that

$$v_i \in L^\infty(0,T; H^m(\Omega)) \cap \mathscr{C}(0,T; H^m(\Omega)), \quad i = 1, \ldots, N.$$

Let us notice that the water-bag solution (42) of the equation

$$\partial_t f + v\partial_z f + \partial_z q(\rho) \cdot \partial_v f = 0,$$

leads to the system

$$\partial_t v_i + \partial_z \left(\frac{v_i^2}{2} - q(\rho)\right) = 0, \quad i = 0, \ldots, N,$$

which cannot be hyperbolic. Indeed, for $q(\rho) = \rho$ and $N = 2$, the system is not hyperbolic since imaginary roots are possible.

The question of the existence of a wide class of solution for equations (43) with general functions q or even with $q(\rho) = \rho$, when the number of bag is infinite, is still an open problem because traditional techniques as averaging lemmas or compensated compactness tools fail. Therefore the water-bag solution (44) could be an interesting way to reach this goal, provided that one is able to pass to the limit with respect to the number of bags in (44).

3.3.2. *The case of an infinite countable number of bag.* As it has been mentioned in the previous remark the existence proof for the quasineutral-MWB when the number of bag is infinite is not an easy task because we need to deal with an infinite dimensional hyperbolic system. We know from theorem 3.4 that the existence time depends on the number of bag. Unfortunately the estimate of the existence time with respect to the number of bag $\mathcal{N}$ leads to a negative result. More precisely we have the following theorem which says that the existence time decreases with the number of bag with a polynomial rate of one half.

Theorem 3.5. *Let assume that $q(\rho) = \rho$ and $0 < c_0 < c_1 < \cdots < c_N$. Moreover we assume that there exists a constant K such that $0 < M_N/m_N < K$ where*

$$M_N = \max\left(c_0^N, \max_{i=1,\cdots,N}(c_i^N - c_{i-1}^N)\right), \quad m_N = \min\left(c_0^N, \min_{i=1,\cdots,N}(c_i^N - c_{i-1}^N)\right).$$

Then there exists a constant $\mathcal{K}$ which depends on K and $\|v_{0i}\|_{H^2(\Omega)}$ such that the maximal existence time T_N of the system (44) satisfies the estimate

$$T_N \le \frac{\mathcal{K}}{\sqrt{1+N}}.$$

Proof. Let us now estimate the existence time with respect to the number of bag N of the system (44) in the case $q(\rho) = \rho$ for the initial data

$$f^0(z,v) = \sum_{i=0}^{N-1} c_i^N \, \mathbb{1}_{v_i^N(0,z) < v < v_{i+1}^N(0,z)} + c_N^N \, \mathbb{1}_{v > v_N^N(0,z)}. \tag{45}$$

We assume that

$$0 < c_0^N < c_1^N < \cdots < c_N^N,$$

in other words the kinetic distribution is increasing with respect to v. The function

$$f_N(t,z,v) = \sum_{i=0}^{N-1} c_i^N \, \mathbb{1}_{v_i^N(t,z)<v<v_{i+1}^N(t,z)} + c_N^N \, \mathbb{1}_{v>v_N^N(t,z)}$$

is a solution of (43) with the initial data (45) if and only if

$$\partial_t v_i^N + \partial_z \left(\frac{(v_i^N)^2}{2} + \rho_N \right) = 0, \quad i = 0, \cdots, N, \tag{46}$$

with the initial data $v_i^N(0,z)$, for $i = 0, \cdots, N$, and with the charge density

$$\rho_N(t,z) = \sum_{i=0}^{N-1} c_i^N (v_{i+1}^N(t,z) - v_i^N(t,z)) + c_N^N(v - v_N^N(t,z)),$$

i.e.

$$\rho_N(t,z) = \int_{-\infty}^{v} f_N(t,z,\xi)\, d\xi,$$

as soon as $v_N^N(t,z) \le v$, with $v > \max_z v_N^N(0,z) + 1$.
Setting

$$V^N = \begin{pmatrix} v_0^N \\ v_1^N \\ \vdots \\ v_{N-1}^N \\ v_N^N \end{pmatrix},$$

then the system can be recast in the symmetric form

$$A_0^N \, \partial_t V^N + A_0^N A^N \, \partial_z V^N = 0,$$

where the matrix A_0^N and A^N are defined by

$$A_0^N = \begin{pmatrix} c_0^N & 0 & \cdots & 0 & 0 \\ 0 & c_1^N - c_0^N & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{N-1}^N - c_{N-2}^N & 0 \\ 0 & 0 & \cdots & 0 & c_N^N - c_{N-1}^N \end{pmatrix},$$

and

$$A^N =$$
$$\begin{pmatrix} v_0^N - c_0^N & c_0^N - c_1^N & \cdots & c_{N-2}^N - c_{N-1}^N & c_{N-1}^N - c_N^N \\ -c_0^N & v_1^N + c_0^N - c_1^N & \cdots & c_{N-2}^N - c_{N-1}^N & c_{N-1}^N - c_N^N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_0^N & c_0^N - c_1^N & \cdots & v_{N-1}^N + c_{N-2}^N - c_{N-1}^N & c_{N-1}^N - c_N^N \\ -c_0^N & c_0^N - c_1^N & \cdots & c_{N-2}^N - c_{N-1}^N & v_N^N + c_{N-1}^N - c_N^N \end{pmatrix}.$$

We notice that

$$
A_0^N\,A^N =
$$

$$
\begin{pmatrix}
c_0^N(v_0^N-c_0^N) & c_0^N(c_0^N-c_1^N) & \cdots & c_0^N(c_{N-1}^N-c_N^N)\\
-c_0^N(c_1^N-c_0^N) & (c_1^N-c_0^N)(v_1^N+c_0^N-c_1^N) & \cdots & (c_1^N-c_0^N)(c_{N-1}^N-c_N^N)\\
\vdots & \vdots & \ddots & \vdots\\
-c_0^N(c_N^N-c_{N-1}^N) & (c_0^N-c_1^N)(c_N^N-c_{N-1}^N) & \cdots & \begin{array}{c}(v_N^N+c_N^N-c_{N-1}^N)\\ .(c_{N-1}^N-c_N^N)\end{array}
\end{pmatrix},
$$

and set

$$
M_N=\max\left(c_0^N,\,\max_{i=1,\cdots,N}(c_i^N-c_{i-1}^N)\right),\quad m_N=\min\left(c_0^N,\,\min_{i=1,\cdots,N}(c_i^N-c_{i-1}^N)\right).
$$

Observing that $M_N\le c_N^N$ and $m_N>0$, we get

$$
m_N\,\mathcal{I}\le A_0^N\le M_N\,\mathcal{I}.
$$

For $\alpha\in\{0,\,1,\,2\}$, the computation of the classical energy of the symmetric system gives

$$
\frac{d}{dt}\langle A_0^N\partial_z^\alpha V^N,\,\partial_z^\alpha V^N\rangle=\langle A_0^N\partial_z A^N\partial_z^\alpha V^N,\,\partial_z^\alpha V^N\rangle+F_\alpha, \tag{47}
$$

with

$$
F_\alpha=\langle A_0^N A^N\partial_z^{\alpha+1}V^N-A_0^N\partial_z^\alpha(A^N\partial_z V^N),\,\partial_z^\alpha V^N\rangle.
$$

Using now Moser type inequalities (proposition 3.7, §3, chapter 13 [60]) and setting $c_{-1}^N=0$, we obtain

$$
\begin{aligned}
|F_\alpha| &= \left|\sum_{i=0}^N(c_i^N-c_{i-1}^N)\sum_{j=0}^N\int_{\mathbb{R}}\left(A_{ij}^N\partial_z^{\alpha+1}V_j^N-\partial_z^\alpha(A_{ij}^N\partial_z V_j^N)\right)\partial_z^\alpha V_i^N\,dz\right|\\
&\le M_N\sum_{i=0}^N\sum_{j=0}^N\left\|A_{ij}^N\partial_z^{\alpha+1}V_j^N-\partial_z^\alpha(A_{ij}^N\partial_z V_j^N)\right\|_{L^2(\Omega)}\left\|\partial_z^\alpha V_i^N\right\|_{L^2(\Omega)}\\
&\le M_N\,C_2\sum_{i=0}^N\sum_{j=0}^N\Big(\left\|\partial_z A_{ij}^N\right\|_{L^\infty(\Omega)}\left\|\partial_z^2 V_j^N\right\|_{L^2(\Omega)}\\
&\qquad+\left\|\partial_z V_j^N\right\|_{L^\infty(\Omega)}\left\|\partial_z^2 A_{ij}^N\right\|_{L^2(\Omega)}\Big)\left\|\partial_z^\alpha V_i^N\right\|_{L^2(\Omega)}.
\end{aligned}
$$

From the form of A^N, we have $\partial_z A_{ij}^N=\delta_{ij}\partial_z V_i^N$, and thus we get

$$
\left\|\partial_z A_{ii}^N\right\|_{L^\infty(\Omega)}\le\left\|\partial_z V_i^N\right\|_{L^\infty(\Omega)},\qquad\left\|\partial_z^2 A_{ii}^N\right\|_{L^2(\Omega)}\le\left\|\partial_z^2 V_i^N\right\|_{L^2(\Omega)}.
$$

Sobolev embedding leading to

$$
\left\|\partial_z V_j^N\right\|_{L^\infty(\Omega)}\le C_1\left\|V_j^N\right\|_{H^2(\Omega)},
$$

we finally obtain

$$
|F_\alpha|\le 2\,M_N\,C_2\,C_1\,\sum_{i=0}^N\left\|V_i^N\right\|_{H^2(\Omega)}^3.
$$

Furthermore the first term of the right hand side of (47) can be estimated as follows

$$\begin{aligned}
\left|\langle A_0^N \partial_z A^N \partial_z^\alpha V^N, \partial_z^\alpha V^N\rangle\right| &= \left|\sum_{i=0}^N (c_i^N - c_{i-1}^N) \sum_{j=0}^N \int_{\mathbb{R}} \partial_z A_{ij}^N \partial_z^\alpha V_j^N \partial_z^\alpha V_i^N \, dz\right| \\
&\leq M_N \left|\sum_{i=0}^N \int_{\mathbb{R}} \partial_z V_i^N (\partial_z^\alpha V_j^N)^2 \, dz\right| \\
&\leq M_N \sum_{i=0}^N \left\|\partial_z V_i^N\right\|_{L^\infty(\Omega)} \left\|\partial_z^\alpha V_i^N\right\|_{L^2(\Omega)}^2 \\
&\leq M_N \, C_1 \sum_{i=0}^N \left\|V_i^N\right\|_{H^2(\Omega)}^3 .
\end{aligned}$$

Therefore the energy estimate (47) gives

$$\begin{aligned}
\langle A_0^N \partial_z^\alpha V^N(t), \partial_z^\alpha V^N(t)\rangle \leq \langle A_0^N \partial_z^\alpha V^N(0), \partial_z^\alpha V^N(0)\rangle \\
+ M_N \, C_1 \, (1 + 2C_2) \int_0^t \sum_{i=0}^N \left\|V_i^N(s)\right\|_{H^2(\Omega)}^3 \, ds,
\end{aligned}$$

and summing over α, for $\alpha \in \{0, 1, 2\}$, it leads to

$$\begin{aligned}
m_N \sum_{i=0}^N \left\|V_i^N(t)\right\|_{H^2(\Omega)}^2 \leq M_N \sum_{i=0}^N \left\|V_i^N(0)\right\|_{H^2(\Omega)}^2 \\
+ 3M_N \, C_1 \, (1 + 2C_2) \int_0^t \sum_{i=0}^N \left\|V_i^N(s)\right\|_{H^2(\Omega)}^3 \, ds. \quad (48)
\end{aligned}$$

If we now introduce the new energy

$$\mathcal{E}_N(t) = \frac{1}{N+1} \sum_{i=0}^N \|V_i^N(t)\|_{H^2(\Omega)}^2, \quad (49)$$

using the estimate (48) then we obtain

$$\mathcal{E}_N(t) \leq \frac{M_N}{m_N} \mathcal{E}_N(0) + 3 \frac{M_N}{m_N} C_1 \, (1 + 2C_2) \frac{1}{N+1} \int_0^t \sum_{i=0}^N \left\|V_i^N(s)\right\|_{H^2(\Omega)}^3 \, ds. \quad (50)$$

Denoting by $F(t)$ the right hand side of (50), we now get

$$
\begin{aligned}
F'(t) &= 3\frac{M_N}{m_n} C_1 (1+2C_2) \frac{1}{N+1} \sum_{i=0}^{N} \left\|V_i^N(t)\right\|_{H^2(\Omega)}^3 \\
&\leq 3\frac{M_N}{m_N} C_1 (1+2C_2) \sqrt{N+1} \left(\frac{1}{(N+1)^2} \sum_{i=0}^{N} \left\|V_i^N(t)\right\|_{H^2(\Omega)}^4 \right)^{1/2} \\
&\quad \left(\frac{1}{N+1} \sum_{i=0}^{N} \left\|V_i^N(t)\right\|_{H^2(\Omega)}^2 \right)^{1/2} \\
&\leq 3\frac{M_N}{m_N} C_1 (1+2C_2) \sqrt{N+1} \left(\frac{1}{N+1} \sum_{i=0}^{N} \left\|V_i^N\right\|_{H^2(\Omega)}^2 (t) \right)^{3/2} \\
&\leq 3\frac{M_N}{m_N} C_1 (1+2C_2) \sqrt{N+1} F(t)^{3/2}.
\end{aligned}
$$

An integration in time leads to

$$
F(t) \leq \frac{1}{\left(\frac{1}{\sqrt{F(0)}} - \frac{3\frac{M_N}{m_N} C_1 (1+2C_2)\sqrt{N+1}t}{2} \right)^2},
$$

and since $F(0) = \frac{M_N}{m_N}\mathcal{E}_N(0)$, we finally get

$$
\mathcal{E}_N(t) \leq \frac{4\mathcal{E}_N(0)\frac{M_N}{m_N}}{\left(2 - 3\sqrt{(N+1)\mathcal{E}_N(0)} \left(\frac{M_N}{m_N}\right)^{3/2} C_1 (1+2C_2)\, t \right)^2}.
$$

Assuming that

$$
0 \leq \frac{M_N}{m_N} \leq K,
$$

an estimate of the existence time T_N is given by

$$
T_N \leq \frac{2}{3\sqrt{(N+1)\mathcal{E}_N(0)}\, K^{3/2} C_1 (1+2C_2)} \leq \frac{\mathcal{K}}{\sqrt{N+1}}.
$$

□

A way to obtain an existence result for the quasineutral-MWB model when the number of the bag is infinite, is to consider a water-bag continuum and a generalized definition of hyperbolicity for integrodifferential hyperbolic system of equations [17].

3.4. **The electromagnetic-MWB model.**

3.4.1. *The case of a finite number of bag.* In this section, we consider the initial value periodic problem

$$
\partial_t v_j^\pm + v_j^\pm \partial_z v_j^\pm + \partial_z \left(\phi + \frac{1}{2}|A_\perp|^2 \right) = 0, \quad v_j^\pm(0,\cdot) = v_{0j}^\pm(\cdot), \quad j = 1,\dots,\mathcal{N}, \tag{51}
$$

$$
-\partial_z^2 \phi = \rho_v - 1, \quad \partial_t^2 A_\perp - \partial_z^2 A_\perp = A_\perp \rho_v, \quad \rho_v = \sum_j^{\mathcal{N}} \mathcal{A}_j (v_j^+ - v_j^-), \tag{52}
$$

with period $\Omega = 1$, $z \in \mathbb{R}/\mathbb{Z}$. Therefore we have the existence theorem.

Theorem 3.6. *(Local classical solution). Assume $v_{0j}^{\pm} \in H^m(\Omega)$ with $m > 3/2$ and $1 \leq j \leq \mathcal{N}$. In addition we suppose that $A_\perp^0 = A_\perp(0,x) \in H^m(\Omega)$ and $A_\perp^1 = (\partial_t A_\perp)(0,x) \in H^{m-1}(\Omega)$. Then for all $\mathcal{N}$ there exists a time $T > 0$ which depends only on $\|v_{0j}^{\pm}\|_{H^m(\Omega)}$, $\|A_\perp^0\|_{H^m(\Omega)}$, $\|A_\perp^1\|_{H^{m-1}(\Omega)}$, $\mathcal{N}$, Ω and $A = \max_{j \leq \mathcal{N}} |\mathcal{A}_j|$ such that the system (51)-(52) admits a unique solution*

$$
\begin{aligned}
&v_j^{\pm},\ A_\perp \in L^\infty(0,T; H^m(\Omega)) \cap \mathrm{Lip}(0,T; H^{m-1}(\Omega)), \quad j = 1, \dots, \mathcal{N}, \\
&\phi \in L^\infty(0,T; H^{m+2}(\Omega)) \cap \mathrm{Lip}(0,T; H^{m+1}(\Omega)).
\end{aligned}
$$

Proof. The proof is based on the Banach's fixed-point theorem. Let us suppose that $(\phi, A_\perp) \in \mathcal{D}_T$ where the set $\mathcal{D}_T$ will be defined further. The goal of the proof is to construct an application $\mathcal{J}$

$$
(\phi, A_\perp) \longrightarrow \left\{ v_{j,\phi,A_\perp}^{\pm} \right\}_{j \leq \mathcal{N}} \longrightarrow \rho_{\phi,A_\perp} \longrightarrow (\widetilde{\phi}, \widetilde{A}_\perp) = \mathcal{J}(\phi, A_\perp)
$$

such that $\mathcal{J}$ leaves invariant the set $\mathcal{D}_T$ and is a contraction. As it has been done in the proof of theorem 3.1 for $j = 1, \dots, \mathcal{N}$, $\alpha \leq m$, we get the energy estimate

$$
\frac{d}{dt}\|v_j^{\pm}\|_{H^\alpha(\Omega)} \leq C(m)\|v_j^{\pm}\|_{H^\alpha(\Omega)}^2 + \|\partial_z^{\alpha+1}\phi\|_{L^2(\Omega)} + \frac{1}{2}\left\|\partial_z^{\alpha+1}|A_\perp|^2\right\|_{L^2(\Omega)}. \tag{53}
$$

Using the interpolation inequality (27) we get

$$
\begin{aligned}
\left\|\partial_z^{\alpha+1}|A_\perp|^2\right\|_{L^2(\Omega)} &= \left\| \sum_{k=0}^{\alpha} \binom{\alpha}{k} \partial_z^{\alpha-k} A_\perp \cdot \partial_z^k(\partial_z A_\perp) \right\|_{L^2(\Omega)} \\
&\leq C(m) \Big(\|A_\perp\|_{L^\infty(\Omega)} \|A_\perp\|_{H^{\alpha+1}(\Omega)} \\
&\qquad + \|\partial_z A_\perp\|_{L^\infty(\Omega)} \|A_\perp\|_{H^\alpha(\Omega)} \Big) \\
&\leq C(m) \|A_\perp\|_{W^{1,\infty}(\Omega)} \|A_\perp\|_{H^{\alpha+1}(\Omega)}.
\end{aligned} \tag{54}
$$

Estimates (53)-(54) and the Sobolev embedding $H^m(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for $m > 3/2$, leads to

$$
\begin{aligned}
\frac{d}{dt}\|\{v_j^{\pm}\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m} &\leq \mathcal{K}_0(m,\mathcal{N}) \left(\|\{v_j^{\pm}\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m}^2 + \|\phi\|_{H^{m+1}(\Omega)} + \|A_\perp\|_{H^{m+1}(\Omega)}^2 \right) \\
&\leq \mathcal{K}_0 \Big(\|\{v_j^{\pm}\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m}^2 \\
&\qquad + \left(1 + \|\phi\|_{H^{m+1}(\Omega)}\right)^2 + \|A_\perp\|_{H^{m+1}(\Omega)}^2 \Big).
\end{aligned} \tag{55}
$$

If we set $X_m(t) = \left(1 + \|\phi(t)\|_{H^m(\Omega)}\right)^2 + \|A_\perp(t)\|_{H^m(\Omega)}^2$, then using the Gronwall Lemma A.1 (see appendix A) we obtain

$$
\|\{v_j^{\pm}(t)\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m} \leq \left(\frac{1}{\|\{v_{0j}^{\pm}\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m} + \mathcal{K}_0 \int_0^t X_{m+1}(s)ds} - \mathcal{K}_0 t \right)^{-1}. \tag{56}
$$

If we use now the d'Alembert formula we can integrate the wave equation for $A_\perp$ to get

$$
\begin{aligned}
\widetilde{A}_\perp(t,z) = \frac{1}{2}(A_\perp^0(z+t) + A_\perp^0(z-t)) + \frac{1}{2}\int_{z-t}^{z+t} A_\perp^1(y)dy \\
+ \frac{1}{2}\int_0^t \int_{z-(t-s)}^{z+(t-s)} (\rho_v \widetilde{A}_\perp)(s,y)dyds.
\end{aligned} \tag{57}
$$

If we apply ∂_z^α to (57) for $\alpha \leq m$, take the L^2-norm, use a Cauchy-Schwarz inequality in time, the Leibniz rules for derivatives, the interpolation inequality (27) and the Sobolev embedding $H^m(\Omega) \hookrightarrow L^\infty(\Omega)$ for $m > 1/2$, we then get

$$\begin{aligned}
&\|\partial_z^\alpha \widetilde{A}_\perp\|_{L^2(\Omega)}^2 \\
\leq\ & 2\|\partial_z^\alpha A_\perp^0\|_{L^2(\Omega)}^2 + 2\|\partial_z^{\alpha-1} A_\perp^1\|_{L^2(\Omega)}^2 \\
& + \left\| \int_0^t \partial_z^{\alpha-1}[(\rho_v \widetilde{A}_\perp)(s, z+t-s) - (\rho_v \widetilde{A}_\perp)(s, z-t+s)]ds \right\|_{L^2(\Omega)}^2 \\
\leq\ & 2\|\partial_z^\alpha A_\perp^0\|_{L^2(\Omega)}^2 + 2\|\partial_z^{\alpha-1} A_\perp^1\|_{L^2(\Omega)}^2 + 2t\int_0^t \left\|\partial_z^{\alpha-1}(\rho_v \widetilde{A}_\perp)\right\|_{L^2(\Omega)}^2 ds \\
\leq\ & 2\|\partial_z^\alpha A_\perp^0\|_{L^2(\Omega)}^2 \\
& +2\|\partial_z^{\alpha-1} A_\perp^1\|_{L^2(\Omega)}^2 + 2t\int_0^t \left\| \sum_{k=0}^{\alpha-1} \begin{pmatrix} \alpha-1 \\ k \end{pmatrix} \partial_z^k \rho_v \partial_z^{\alpha-1-k} \widetilde{A}_\perp \right\|_{L^2(\Omega)}^2 ds \\
\leq\ & 2\|\partial_z^\alpha A_\perp^0\|_{L^2(\Omega)}^2 \\
& +2\|\partial_z^{\alpha-1} A_\perp^1\|_{L^2(\Omega)}^2 + C(\alpha)t\int_0^t \|\rho_v\|_{H^{\alpha-1}(\Omega)}^2 \|\widetilde{A}_\perp\|_{H^{\alpha-1}(\Omega)}^2 ds.
\end{aligned} \tag{58}$$

The regularity properties of the solution of the Poisson equation in L^2 imply that

$$\|\widetilde{\phi}\|_{H^m(\Omega)} \leq C(\Omega)\, \|\rho_v\|_{H^{\max\{m-2,0\}}(\Omega)} \leq \mathcal{K}_3(\Omega, A) \|\{v_j^\pm\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^{\max\{m-2,0\}}}. \tag{59}$$

Using estimates (56), (58) and (59) we obtain

$$\begin{aligned}
\widetilde{X}_m(t) \quad \leq \quad & \|A_\perp^0\|_{H^m(\Omega)}^2 + \|A_\perp^1\|_{H^{m-1}(\Omega)}^2 + \left(1 + C(\Omega)\, \|\rho_v(t)\|_{H^{m-2}(\Omega)}\right)^2 \\
& +C(m)t\int_0^t \|\rho_v(s)\|_{H^{m-1}(\Omega)}^2 \|\widetilde{A}_\perp(s)\|_{H^{m-1}(\Omega)}^2 ds \\
\leq\ & \mathcal{K}_2 + 2 + 2\mathcal{K}_3^2(\Omega, A)\left(\frac{1}{\mathcal{K}_1 + \mathcal{K}_0\int_0^t X_m(s)ds} - \mathcal{K}_0 t\right)^{-2} \\
& +t\int_0^t \mathcal{K}_4(m, \Omega, A)\left(\frac{1}{\mathcal{K}_1 + \mathcal{K}_0\int_0^s X_m(\tau)d\tau} - \mathcal{K}_0 s\right)^{-2} \widetilde{X}_m(s)ds,
\end{aligned} \tag{60}$$

where we have set

$$\mathcal{K}_1 = \|\{v_{0j}^\pm\}_{j\leq\mathcal{N}}\|_{\mathbb{H}^m} \quad \text{and} \quad \mathcal{K}_2 = \|A_\perp^0\|_{H^m(\Omega)}^2 + \|A_\perp^1\|_{H^{m-1}(\Omega)}^2.$$

Let us now define the set $\mathcal{D}_T$ as

$$\begin{aligned}
\mathcal{D}_T = \Big\{ & (\phi, A_\perp) \ \mid\ \phi,\ A_\perp \in L^\infty(0, T; H^m(\Omega)), \\
& (1 + \|\phi\|_{L^\infty(0,T;H^m(\Omega))})^2 + \|A_\perp\|_{L^\infty(0,T;H^m(\Omega))}^2 < \mathcal{K}[\mathcal{K}_2 + 2 + 2(\mathcal{K}_1\mathcal{K}_3)^2] \Big\}
\end{aligned}$$

where $\mathcal{K} > 1$ is a purely numerical constant. Using the estimate (60), a Gronwall inequality show that $\widetilde{X}_m \leq \mathcal{K}[\mathcal{K}_2 + 2 + 2(\mathcal{K}_1\mathcal{K}_3)^2]$ for all $t \in [0, T]$, T small enough.

Therefore there exists a time $T > 0$ such that the application $\mathcal{J}$ maps $\mathcal{D}_T$ into itself. From the multi-water-bag equations (51) we have $v_j^\pm \in L^\infty(0, T; H^m(\Omega)) \cap \mathrm{Lip}(0, T; H^{m-1}(\Omega))$ for $1 \leq j \leq \mathcal{N}$.

We now need to prove that $\mathcal{J}$ is a contraction to get a unique solution. To this purpose we must evaluate the difference

$$(\widetilde{\phi}_1 - \widetilde{\phi}_2, \widetilde{A}_{1\perp} - \widetilde{A}_{2\perp}) = \mathcal{J}(\phi_1, A_{1\perp}) - \mathcal{J}(\phi_2, A_{2\perp})$$

where $\phi_1,\ \phi_2,\ A_{1\perp}$, and $A_{2\perp}$ belong to $\mathcal{D}_T$. If we set $v_j^\pm = v_{1j}^\pm - v_{2j}^\pm$, $\phi = \phi_1 - \phi_2$, $A_\perp = A_{1\perp} - A_{2\perp}$, $\widetilde{\phi} = \widetilde{\phi}_1 - \widetilde{\phi}_2$, $\widetilde{A}_\perp = \widetilde{A}_{1\perp} - \widetilde{A}_{2\perp}$ and if we substract equations (51)-(52) for each solution we obtain the system, for $j = 1, \dots, \mathcal{N}$,

$$\partial_t v_j^\pm + v_j^\pm \partial_z v_{1j}^\pm + v_{2j}^\pm \partial_z v_j^\pm + \partial_z \left(\phi + \frac{1}{2}(A_{1\perp} \cdot A_\perp + A_{2\perp} \cdot A_\perp) \right) = 0, \tag{61}$$

$$-\partial_z^2 \phi = \rho_v, \quad \rho_v = \sum_j^{\mathcal{N}} \mathcal{A}_j (v_j^+ - v_j^-), \quad v_j^\pm(0,\cdot) = 0, \quad j = 1, \dots, \mathcal{N},$$

$$\partial_t^2 A_\perp - \partial_z^2 A_\perp = A_\perp \rho_{1v} + A_{2\perp} \rho_v, \quad A_\perp^0 = 0, \quad A_\perp^1 = 0. \tag{62}$$

Following the proof of theorem 3.1, equation (61) leads to

$$\frac{d}{dt} \|\{v_j^\pm(t)\}_{j \le \mathcal{N}}\|_{\mathbb{H}^{m-1}} \le C_1(m, \mathcal{N}, \mathcal{K}, \{\mathcal{K}_i\}_{i=1,2,3}) \left(\|\{v_j^\pm(t)\}_{j \le \mathcal{N}}\|_{\mathbb{H}^{m-1}} + \sqrt{2} Y_m^{1/2}(t) \right). \tag{63}$$

where $Y_m(t) = \|\phi(t)\|^2_{H^m(\Omega)} + \|A_\perp(t)\|^2_{H^m(\Omega)}$. Using the Gronwall Lemma A.1 we get

$$\|\{v_j^\pm(t)\}_{j \le \mathcal{N}}\|_{\mathbb{H}^{m-1}} \le \sqrt{2} C_1 e^{C_1 t} \int_0^t Y_m^{1/2}(s) ds. \tag{64}$$

Following the proof of the estimate (58), using equation (62) and estimate (59) we get

$$\|\widetilde{A}_\perp(t)\|^2_{H^{m-1}(\Omega)} \le C_2(m, \mathcal{K}, \{\mathcal{K}_i\}_{i=1,2,3}) t \int_0^t \|\{v_j^\pm(s)\}_{j \le \mathcal{N}}\|^2_{\mathbb{H}^{m-2}} + \|\widetilde{A}_\perp(s)\|^2_{H^{m-2}(\Omega)} ds. \tag{65}$$

Using estimates (64)-(65) and (59) we get

$$\widetilde{Y}_{m-1}(t) \le \left(\mathcal{K}_3 C_1 \sqrt{2} \int_0^t Y_{m-1}^{1/2}(s) ds \right)^2 e^{2C_1 t} + C_2 t \int_0^t \left[\left(C_1 \sqrt{2} \int_0^s Y_{m-1}^{1/2}(\tau) d\tau \right)^2 e^{2C_1 s} + \widetilde{Y}_{m-1}(s) \right] ds. \tag{66}$$

Once again, a Cauchy-Schwarz inequality and a Gronwall lemma shows that $\mathcal{J}$ is a contraction provided that T is small enough. □

3.4.2. *The case of an infinite countable number of bag.* The theorem 3.6 is not true for an infinite number of bag because the constants involving in the proof depend on the number of bag. Nevertheless the theorem 3.6 can be extended to the case of an infinite countable number of bag by replacing the norm $\|\cdot\|_{\mathbb{H}^m}$ with the norm $\|\cdot\|_{\mathrm{L}^1 \cap \infty \mathrm{H}^m}$ given by the definition (32). Therefore we have the existence theorem.

Theorem 3.7. *(Local classical solution). Assume $v_{0j}^\pm \in H^m(\Omega)$ with $m > 3/2$ and $j \in \mathbb{N}^*$. Let us suppose that the coefficients $\{\mathcal{A}_j\}_{j \in \mathbb{N}^*}$ are such that the sum $\sum_{j \in \mathbb{N}^*} |\mathcal{A}_j| = \mathcal{A}$ is bounded. In addition we suppose that $A_\perp^0 = A_\perp(0,x) \in H^m(\Omega)$ and $A_\perp^1 = (\partial_t A_\perp)(0,x) \in H^{m-1}(\Omega)$. Then there exists a time $T > 0$ which depends* *only on* $\|v_{0j}^{\pm}\|_{H^m(\Omega)}$, $\|A_\perp^0\|_{H^m(\Omega)}$, $\|A_\perp^1\|_{H^{m-1}(\Omega)}$, $\mathcal{A}$ *and* Ω *such that the system (51)-(52) admits a unique solution*

$$v_j^{\pm},\ A_\perp \in L^\infty(0,T;H^m(\Omega)) \cap \mathrm{Lip}(0,T;H^{m-1}(\Omega)), \quad j \in \mathbb{N}^*,$$
$$\phi \in L^\infty(0,T;H^{m+2}(\Omega)) \cap \mathrm{Lip}(0,T;H^{m+1}(\Omega)).$$

Proof. Following the lines of the proof of theorem 3.6, we obtain the same estimates (60) and (66) where the norm $\|\cdot\|_{\mathbb{L}^{1\cap\infty}\mathbb{H}^m}$ can be subtituted for the norm $\|\cdot\|_{\mathbb{H}^m}$. Therefore we can construct a set $\mathcal{D}_T$ and an application $\mathcal{J}$ which maps $\mathcal{D}_T$ into itself and is also a contraction. The Banach's fixed-point theorem then yields the local existence and uniqueness result. □

3.4.3. *The case of a continuum of bag.* In order to consider a continuum of bag, as in section 3.2.3, we consider two Lagrangian foliations to be the families of sheets $v^\pm = v^\pm(t,z,a)$ labelled by the Lagrangian label $a \in [0,1]$ where the water-bag continuum $v^\pm(t,z,a)$ are smooth functions. The system (51)-(52) is still valid if we replace the counting measure by the Lebesgue measure da, which means that the water-bag continuum $v^\pm$ satisfy the continuous water-bag model given by

$$\partial_t v^\pm + v^\pm \partial_z v^\pm + \partial_z\left(\phi + \frac{1}{2}|A_\perp|^2\right) = 0, \quad v^\pm(t=0) = v_0^\pm, \tag{67}$$

$$-\partial_z^2\phi = \rho_v - 1, \quad \partial_t^2 A_\perp - \partial_z^2 A_\perp = A_\perp \rho_v, \quad \rho_v = \int_0^1 (v^+ - v^-)da. \tag{68}$$

with period $\Omega = 1$, $z \in \mathbb{R}/\mathbb{Z}$, $a \in [0,1]$. Therefore we have the following existence theorem.

Theorem 3.8. *(Local classical solution). Assume* $v_0^\pm \in H^m(\mathcal{D})$ *with* $m > 2$ *and* $\mathcal{D} = \Omega \times [0,1]$. *In addition we suppose that* $A_\perp^0 = A_\perp(t=0) \in H^m(\mathcal{D})$ *and* $A_\perp^1 = (\partial_t A_\perp)(t=0) \in H^{m-1}(\mathcal{D})$. *Then there exists a time* $T > 0$ *which depends only on* $\|v_0^\pm\|_{H^m(\mathcal{D})}$, $\|A_\perp^0\|_{H^m(\Omega)}$, $\|A_\perp^1\|_{H^{m-1}(\Omega)}$, *and* $\mathcal{D}$ *such that the system (67)-(68) admits a unique solution*

$$v^\pm \in L^\infty(0,T;H^m(\mathcal{D})) \cap \mathrm{Lip}(0,T;H^{m-1}(\mathcal{D})),$$
$$\phi \in L^\infty(0,T;H^{m+2}(\Omega) \cap \mathrm{Lip}(0,T;H^{m+1}(\Omega)),$$
$$A_\perp \in L^\infty(0,T;H^m(\Omega)) \cap \mathrm{Lip}(0,T;H^{m-1}(\Omega)).$$

Proof. The proof of the theorem is the same as theorem 3.6, except that the problem is now set in a two-dimensional space, which implies more regularity for the initial conditions because of the Sobolev embeddings. □

4. **Numerical approximation.** In this section we consider a periodic plasma of period L, $z \in \Omega =]0,L[$. After the normalization of the equations (7), (10), (12) and (16)-(18) the multi-water-bag equations are readily obtained

$$\partial_t v_j^\pm + v_j^\pm \partial_z v_j^\pm + \partial_z\left(\phi + \frac{1}{2}|A_\perp|^2\right) = 0, \tag{69}$$

together with the Poisson equation

$$-\partial_z^2\phi = \sum_{j\le\mathcal{N}} \mathcal{A}_j(v_j^+ - v_j^-) - n_0, \tag{70}$$

the quasi-neutral equation for the ion acoustic waves

$$\phi = \frac{Z_i}{n_0\tau}\Big(Z_i \sum_{j\le\mathcal{N}} \mathcal{A}_j(v_j^+ - v_j^-) - n_0\Big), \tag{71}$$

the waves equation for electromagnetic laser-plasma interaction

$$\partial_t^2 A_\perp - \partial_z^2 A_\perp = A_\perp \sum_{j \le \mathcal{N}} \mathcal{A}_j (v_j^+ - v_j^-), \tag{72}$$

with Z_i the number of charge and $\tau = T_i/T_e$. Moreover we add the initial conditions $v_j^\pm(0,\cdot) = v_{0j}^\pm(\cdot)$.

4.1. **Numerical method.** In this section we present briefly the numerical method we use to solve the equations (69)-(70) with $A_\perp = 0$, (69) and (71) with $A_\perp = 0$ and the system formed by the equations (69)-(70) and (72). The discontinuous Galerkin (DG) method [26, 30] has been used to investigate these equations. This is a finite element method space discretization by discontinuous approximations, that incorporates the ideas of numerical fluxes and slope limiters used in high-order finite difference and finite volume schemes. The DG methods can be combined with Runge-Kutta or Lax-Wendroff time discretization scheme to give stable, high-order accurate, highly parallelizable schemes that can easily handle h-p adaptivity, complicated geometries and boundaries conditions.

Let us note that the way we apply the DG method is original for our model, because strictly speaking the multi-water-bag model (39) is a system of conservation laws and should be solved using DG schemes for one-dimensional system [25] resulting in computing eigenvalues of the jacobian matrix. Nevertheless, solving as we do each half-bag separately as a scalar conservation law still works. From the numerical point of view it is very interesting because we do not have eigenvalue problems to solve and even better the parallel computation with respect to the bag number can be performed easily. This remark will be very useful for gyrokinetic applications [45, 54, 55, 15].

4.1.1. *Discretization of the multi-water-bag equations.* Let be Ω the domain of computation and $\mathcal{M}_h$ a partition of Ω of element K such that $\cup_{K\in\mathcal{M}_h}\overline{K} = \overline{\Omega}$, $K \cap Q = \emptyset$, $K, Q \in \mathcal{M}_h$, $K \neq Q$. We set $h = \max_{K\in\mathcal{M}_h} h_K$ where h_K is the exterior diameter of a finite element K. The first step of the method is to write the equations (69) in a variational form on any element K of the partition $\mathcal{M}_h$. Using a Green formula, for any enough regular test-function φ, for all $j = 1, \ldots, \mathcal{N}$, we get

$$\int_K \partial_t v_j^\pm \varphi dz - \int_K \left(f(v_j^\pm) + \phi(z) + \frac{1}{2}|A_\perp|^2 \right) \partial_z \varphi \, dz \\ + \int_{\partial K} \left(f(v_j^\pm) + \phi(z) + \frac{1}{2}|A_\perp|^2 \right) n_K \varphi \, d\Gamma, \quad \forall\, K \in \mathcal{M}_h, \tag{73}$$

where ∂K denotes the boundary of K, n_K denotes the outward unit normal to ∂K, and $f(\cdot) = (\cdot)^2/2$.

We now seek an approximate solution $(v_{h,j}^\pm, \phi_h, A_{\perp\, h})$ whose restriction to the element K of the partition $\mathcal{M}_h$ of Ω belongs, for each value of the time variable, to the finite dimensional local space $\mathscr{P}(K)$, typically a space of polynomials. Therefore we set

$$\mathscr{P}_h(\Omega) = \left\{ \psi \;\mid\; \psi_{|_K} \in \mathscr{P}(K), \forall\, K \in \mathcal{M}_h \right\}.$$

We now determine the approximate solution $(v_{h,j}^\pm, \phi_h, A_{\perp\, h})_{|_K} \in \mathscr{P}(K)^{\otimes^4}$ for $t > 0$, on each element K of $\mathcal{M}_h$ by imposing that, for all $\varphi_h \in \mathscr{P}(K)$, for all

$j = 1, \ldots, \mathcal{N}$,

$$\int_K \partial_t v^{\pm}_{h,j} \varphi_h dz - \int_K \left(f(v^{\pm}_{h,j}) + \phi_h(z) + \frac{1}{2}|A_{h,\perp}(z)|^2 \right) \partial_z \varphi_h \, dz + \int_{\partial K} \left(\widehat{f\, n_K}(v^{\pm}_{h,j}) + \widehat{\phi_h\, n_K} + \frac{1}{2}\widehat{|A_{h,\perp}|^2\, n_K} \right) \varphi_h \, d\Gamma, \quad (74)$$

where we have replaced the flux terms $(f(v^{\pm}_j) + \phi + \frac{1}{2}|A_{\perp}|^2)\, n_K$ in (73) by the numerical flux $\widehat{f\, n_K}(v^{\pm}_{h,j}) + \widehat{\phi_h\, n_K} + \frac{1}{2}\widehat{|A_{h,\perp}|^2\, n_K}$ because in (73) the term arising from the boundary of the cell K are not well defined or have no sense since $v^{\pm}_{h,j}$, ϕ_h, $A_{h,\perp}$ and φ_h are discontinuous (by construction of the space of approximation) on the boundary ∂K of the element K. It now remains to define the numerical flux $\widehat{f\, n_K} + \widehat{\phi_h\, n_K} + \frac{1}{2}\widehat{|A_{h,\perp}|^2\, n_K}$. For two adjacent cells K^+ and K^- of $\mathcal{M}_h$ and a point z of their common boundary at which the vector $n_{K^\pm}$ are defined, we set $\varphi^{\pm}_h(z) = \lim_{\epsilon \to 0} \varphi_h(z - \epsilon n_{K^\pm})$ and call these values the traces of φ_h from the interior of $K^\pm$. Therefore the numerical flux at z is a function of the traces $v^{\pm,\pm}_{h,j}$, i.e.

$$\widehat{f\, n_{K^-}}(v^{\pm}_{h,j})(z) = \widehat{f\, n_{K^-}}(v^{\pm,-}_{h,j}(z), v^{\pm,+}_{h,j}(z)).$$

Besides the numerical flux must be consistent with the non linearity $f\, n_{K^-}$, which means that we should have $\widehat{f\, n_{K^-}}(v, v) = f(v)\, n_{K^-}$. In order to get monotone scheme in case of piecewise-constant approximation the numerical flux must be conservative, i.e

$$\widehat{f\, n_{K^-}}(v^{\pm,-}_{h,j}(z), v^{\pm,+}_{h,j}(z)) + \widehat{f\, n_{K^+}}(v^{\pm,+}_{h,j}(z), v^{\pm,-}_{h,j}(z)) = 0$$

and the mapping $v \mapsto \widehat{f\, n_{K^-}}(v, \cdot)$ must be non-decreasing. There exists several examples of numerical fluxes satisfying the above requirements: the Godunov flux, the Engquist-Osher flux, the Lax-Friedrichs flux (see [26]). For the numerical flux $\widehat{\phi_h\, n_{K^-}}$ and $\widehat{|A_{h,\perp}|^2\, n_{K^-}}$ we can choose average, left or right flux. We can also choose other numerical fluxes [30, 26]. Therefore, for each cell K, after the space-discretization step, we get the ordinary differential equation (ODE)

$$\mathfrak{M} \frac{d}{dt} v^{\pm}_{h,j_{|K}} = \mathcal{L}_K \left(\left\{ v^{\pm}_{h,j_{|K'}}, \phi_{h_{|K'}}, A_{h,\perp_{|K'}} \mid \overline{K'} \cap \overline{K} \in \partial K \right\} \right), \quad (75)$$

$\forall K \in \mathcal{M}_h$ and $j = 1, \ldots, \mathcal{N}$. In the general case, the local mass matrix $\mathfrak{M}$ of low order (equal to the dimension of the local space $\mathscr{P}(K)$) is easily invertible. If we choose orthogonal polynomials $\mathfrak{M}$ is diagonal. Here we take the Legendre polynomials as L^2-orthogonal basis function. Our code can run with Legendre polynomial to any order, but for the numerical results exposed in the next section we choose $n = 2$, i.e. polynomial of degree two.
Therefore we have to solve the ODE

$$\frac{d}{dt} v^{\pm}_{h,j} = \mathscr{L}_h \left(v^{\pm}_{h,j}, \phi_h, A_{h,\perp} \right), \quad j = 1, \ldots, \mathcal{N}. \quad (76)$$

In order to solve (76) we can use Runge-Kutta methods [39]. For numerical stability considerations we have to choose $k+1$ stage Runge-Kutta method of order $k+1$ for DG discretizations using polynomials of degree k if we do not want our CFL number to be too small. As we take polynomial of degree two we choose a the third-order strong stability-preserving Runge-Kutta method [39]: for all $1 \leq j \leq \mathcal{N}$

$$\begin{aligned} v^{\pm}_{h,j}(t_1) &= v^{\pm}_{h,j}(t^n) + \Delta t\, \mathscr{L}_h \left(v^{\pm}_{h,j}(t^n), \phi_h(t^n), A_{h,\perp}(t^n)\right), \\ v^{\pm}_{h,j}(t_2) &= \frac{3}{4} v^{\pm}_{h,j}(t^n) + \frac{1}{4} v^{\pm}_{h,j}(t_1) + \frac{1}{4}\Delta t\, \mathscr{L}_h \left(v^{\pm}_{h,j}(t_1), \phi_h(t_1), A_{h,\perp}(t_1)\right), \\ v^{\pm}_{h,j}(t^{n+1}) &= \frac{1}{3} v^{\pm}_{h,j}(t^n) + \frac{2}{3} v^{\pm}_{h,j}(t_2) + \frac{2}{3}\Delta t\, \mathscr{L}_h \left(v^{\pm}_{h,j}(t_2), \phi_h(t_2), A_{h,\perp}(t_2)\right), \end{aligned}$$

with $t^n = n\Delta T$, $\Delta t = T/N_T$, and t_1 and t_2 time between t^n and t^{n+1}.

For the discretization of the initial condition we take $v^{\pm}_{0h,j}$ on the cell K to be the L^2-projection of $v^{\pm}_{0j}(\cdot)$ on $\mathscr{P}(K)$, i.e for all $\varphi_h \in \mathscr{P}(K)$

$$\int_K v^{\pm}_{0h,j} \varphi_h \, dz = \int_K v^{\pm}_{0j} \varphi_h \, dz, \quad \forall\, K \in \mathcal{M}_h.$$

4.1.2. *Discretization of the quasineutral equation.* For solving the equation (71) we take its L^2-projection on $\mathscr{P}(K)$, i.e for all $\varphi_h \in \mathscr{P}(K)$

$$\int_K \phi_h \varphi_h \, dz = \int_K \varphi_h \frac{Z_i}{n_0 \tau} \left(Z_i \sum_{j=1}^{\mathcal{N}} A_j (v^+_{h,j} - v^-_{h,j}) - n_0 \right) dz, \quad \forall\, K \in \mathcal{M}_h.$$

4.1.3. *Discretization of the Poisson equation.* We aim now at solving the Poisson equation (70). In order to obtain a stable and consistent global scheme we solve the Poisson equation in the framework of local discontinuous Galerkin method. Using Green formula we can rewrite the problem (70) in the following variational form: find $E_h \in \mathscr{P}_h(\Omega)$ and $\phi_h \in \mathscr{P}_h(\Omega)$ such that for all φ_h, $\psi_h \in \mathscr{P}_h(\Omega)$, for all $K \in \mathcal{M}_h$

$$\int_K E_h \varphi_h \, dz = \int_K \phi_h \partial_z \varphi_h \, dz - \int_{\partial K} \widehat{\phi}_h \varphi_h n_{K^-} \, d\Gamma, \tag{77}$$

$$\int_K E_h \partial_z \psi_h \, dz = \int_{\partial K} \widehat{E}_h n_{K^-} \psi_h \, d\Gamma - \int_K \rho_h \psi_h \, dz, \tag{78}$$

where E_h is an approximation of $E = -\partial_z \phi$, and ρ_h stands for the right hand side of (70) where we have replaced $v^{\pm}_j$ by their approximations $v^{\pm}_{h,j}$. If we set n the outward unit normal to $\partial\Omega$, $\mathcal{E}^{\circ}_h$ the set of interior edges of $\mathcal{M}_h$, $\mathcal{E}^{\partial}_h$ the set of boundary edges of $\mathcal{M}_h$ and if we use the notations $[\varphi_h] = \varphi^+_h n_{K^-} + \varphi^-_h n_{K^+}$, $\{\varphi_h\} = \frac{1}{2}(\varphi^+_h + \varphi^-_h)$, then we have

$$\sum_{K \in \mathcal{M}_h} \int_{\partial K} \psi_{K^-} \varphi_{K^-} n_{K^-} \, d\Gamma = \int_{\mathcal{E}^{\circ}_h} ([\psi]\{\varphi\} + [\varphi]\{\psi\}) \, d\Gamma + \int_{\mathcal{E}^{\partial}_h} \psi \varphi \, n \, d\Gamma. \tag{79}$$

If we take $\varphi_h = E_h$ in (77), $\psi_h = \phi_h$ in (78), summing over the cell K and using (79) we obtain

$$\mathcal{R}_h + \int_\Omega |E_h|^2 \, dz = \int_\Omega \rho_h \phi_h \, dz, \tag{80}$$

where

$$\mathcal{R}_h = \int_{\mathcal{E}^{\circ}_h} \left(\{\widehat{E}_h - E_h\}[\phi_h] + \{\widehat{\phi}_h - \phi_h\}[E_h] \right) d\Gamma + \int_{\mathcal{E}^{\partial}_h} \left(\phi_h(\widehat{E}_h - E_h) + \widehat{\phi}_h E_h \right) n \, d\Gamma. \tag{81}$$

Let us now choose the numerical fluxes as follows

$$\begin{aligned}&\widehat{E}_h = \{E_h\} + \alpha_{11}[\phi_h] + \alpha_{12}[E_h], \quad \widehat{\phi}_h = \{\phi_h\} - \alpha_{11}[\phi_h] + \alpha_{22}[E_h] \quad \text{on } \mathcal{E}_h^\circ\\ &\widehat{E}_h = E_h + \alpha_{11}\phi_h n, \quad \widehat{\phi}_h = 0, \quad \text{on } \mathcal{E}_h^\partial,\end{aligned} \tag{82}$$

where $\alpha_{11} > 0$, $\alpha_{22} \geq 0$ and α_{12} is an arbitrary real number. Pluging (82) into (81) yields

$$\mathcal{R}_h = \int_{\mathcal{E}_h^\circ} (\alpha_{11}[\phi_h]^2 + \alpha_{22}[E_h]^2)\, d\Gamma + \int_{\mathcal{E}_h^\partial} \alpha_{11}|\phi_h|^2\, d\Gamma \geq 0. \tag{83}$$

If we set $\rho_h = 0$ then we get $[\phi_h] = 0$, ${\phi_h}_{|_{\mathcal{E}_h^\partial}} = 0$ and $E_h = 0$. Therefore the equation (77) can be rewritten as

$$\int_K \psi \partial_z \phi_h\, dz = 0, \quad \forall\, \psi \in \mathscr{P}(K), \quad \forall\, K \in \mathcal{M}_h,$$

which means that $\phi_h = 0$ on $\overline{\Omega}$ so that the approximate solution ϕ_h is well defined. Now our method can afford a unique approximate solution. It can easily be computed as follows. If we take the equation (77), sum over the cell K, by using (79) we get

$$a(E_h, \varphi_h) - b(\phi_h, \varphi_h) = 0, \quad \forall\, \varphi_h \in \mathscr{P}_h(\Omega), \tag{84}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are

$$\begin{aligned} a(u,v) &= \int_\Omega uv\, dz + \int_{\mathcal{E}_h^\circ} \alpha_{22}[u][v]\, d\Gamma,\\ b(w,u) &= \int_\Omega \partial_z uw\, dz + \int_{\mathcal{E}_h^\circ} [u](\alpha_{12}[w] - \{w\})\, d\Gamma.\end{aligned}$$

Using integration by part we get

$$-\int_K E_h \partial_z \varphi_h\, dz = -\int_{\partial K} E_h n_{K^-} \varphi_h\, d\Gamma + \int_K \partial_z E_h \varphi_h\, dz. \tag{85}$$

If we add (78) to (85), sum over all cell K, and use (79) then we get

$$b(\psi_h, E_h) + c(\psi_h, \phi_h) = F(\psi_h), \quad \forall\, \psi_h \in \mathscr{P}_h(\Omega), \tag{86}$$

where the bilinear form $c(\cdot,\cdot)$ and the linear form $F(\cdot)$ are

$$c(w,p) = \alpha_{11}\int_{\mathcal{E}_h^\circ} [w][p]\, d\Gamma + \alpha_{11}\int_{\mathcal{E}_h^\partial} pw\, d\Gamma, \quad F(w) = \int_\Omega w\rho_h\, dz.$$

The variational formulation (84) and (86), leads to the matrix formulation

$$\Phi_h^T \mathcal{A} E_h - \phi_h^T \mathcal{B}\Psi_h = 0, \quad \Psi_h^T \mathcal{B} E_h - \Psi_h^T \mathcal{C}\phi_h = \Psi_h^T F_h, \quad \forall\, \Psi_h,\, \Phi_h,$$

which is equivalent to solve the linear system

$$E_h = \mathcal{A}^{-1}\mathcal{B}^T \phi_h, \quad (\mathcal{B}\mathcal{A}^{-1}\mathcal{B}^T + \mathcal{C})\phi_h = F_h. \tag{87}$$

Equation (87) can be solved by direct (LU decomposition for example) or iterative methods (Conjugate gradient for example) of linear algebra. Let us note that if $\alpha_{22} = 0$, then the matrix $\mathcal{A}$ is diagonal by block, and therefore it is easier to invert.

4.1.4. *Discretization of the waves equation.* The remaining task is now to solve the waves equation (72). To this purpose we rewrite the equation (72). Introducing the propagator fields $E^{\pm} = E_{\perp 1} \pm B_{\perp 2}$ and $F^{\pm} = E_{\perp 2} \pm B_{\perp 1}$, the wave equation (72) is equivalent to the system

$$\partial_t E^{\pm} \pm \partial_x E^{\pm} = -J_{\perp 1}, \quad \partial_t F^{\pm} \mp \partial_x F^{\pm} = -J_{\perp 2}, \tag{88}$$

where for $i = 1, 2$,

$$J_{\perp i} = -A_{\perp i} \sum_{j \leq \mathcal{N}} \mathcal{A}_j (v_j^+ - v_j^-),$$

and

$$\partial_t A_{\perp 1} = -E_{\perp 1} = \frac{1}{2}(E^+ + E^-), \quad \partial_t A_{\perp 2} = -E_{\perp 2} = \frac{1}{2}(F^+ + F^-).$$

Let us start with equations (88). After writting equations (88) in variational forms on any element K of the partition $\mathcal{M}_h$ by using a Green formula, we determine the approximate solution $(E_h^{\pm}, F_h^{\pm})_{|K} \in \mathscr{P}(K)^{\otimes^4}$ for $t > 0$, on each element K of $\mathcal{M}_h$ by imposing for all $\varphi_h \in \mathscr{P}(K)$,

$$\int_K \partial_t E_h^{\pm} \varphi_h dz \mp \int_K E_h^{\pm} \partial_z \varphi_h \, dz \pm \int_{\partial K} \widehat{E_h^{\pm} \, n_K} \varphi_h \, d\Gamma = -\int_K J_{h,\perp 1} \varphi_h \, dz, \tag{89}$$

$$\int_K \partial_t F_h^{\pm} \varphi_h dz \pm \int_K F_h^{\pm} \partial_z \varphi_h \, dz \mp \int_{\partial K} \widehat{F_h^{\pm} \, n_K} \varphi_h \, d\Gamma = -\int_K J_{h,\perp 2} \varphi_h \, dz, \tag{90}$$

where upwind fluxes are chosen for the numerical quantities $\widehat{E_h^{\pm} \, n_{K^-}}$ and $\widehat{F_h^{\pm} \, n_{K^-}}$. The discontinuous-Galerkin projection of the equation $\partial_t A_{\perp} = -E_{\perp}$ is simply

$$\int_K \partial_t A_{h,\perp} \varphi_h dz = -\int_K E_{h,\perp} \varphi_h \, dz. \tag{91}$$

As in section 4.1.1, the equations (89)-(91) leads to the ODEs

$$\frac{d}{dt} \mathcal{F}_h = \mathcal{J}_h \left(\mathcal{F}_h, \left\{ v_{h,j}^{\pm} \right\}_{j=1,\dots,\mathcal{N}} \right), \tag{92}$$

where the compact notation $\mathcal{F}_h = (E_h^{\pm}, F_h^{\pm}, A_{h,\perp})$ has been used for the electromagnetic fields. The ODEs sytem (92) is solved by a third-order strong stability-preserving Runge-Kutta method [39], as in section 4.1.1.

5. Numerical results.

5.1. **Construction of a multi-water-bag equilibrium.** This section is devoted to the construction of the initial conditions which will be used to initialize the numerical schemes depicted in the previous section. The initial condition is constructed as a perturbation of an homogeneous equilibrium. Let us construct first an equilibrium. To this purpose we consider an homogeneous equilibrium distribution function $f_0(v)$. For simplicity reason we suppose f_0 is an even function of v (odd momenta are zero). In the mult-water-bag formalism it means symmetrical equilibrium contours $\pm v_{0j}$, $1 \leq j \leq \mathcal{N}$. Let us define the ℓ-moment, $\mathscr{M}_\ell$, of f_0 (ℓ even only)

$$\mathscr{M}_\ell(f_0) = \int_{-\infty}^{\infty} v^\ell \, f_0(v) \, dv \tag{93}$$

and the ℓ-moment of the corresponding multi-water-bag

$$\mathscr{M}_\ell(\text{MWB}) = \frac{1}{\ell + 1} \sum_{j=1}^{\mathcal{N}} 2\mathcal{A}_j \, v_{0j}^{\ell+1}. \tag{94}$$

Let us now sample the v-axis with appropriate v_{0j}'s. Thus equating equations (93) and (94) for $\ell = 0, 2, \ldots, 2(\mathcal{N}-1)$ yields a system of $\mathcal{N}$ equations for the $\mathcal{N}$ unknown $\mathcal{A}_j$, $j = 1, \ldots, \mathcal{N}$. Using an integration by parts we get

$$\sum_{j=1}^{\mathcal{N}} 2\mathcal{A}_j\, v_{0j}^{\ell+1} = -\int_{-\infty}^{\infty} v^{\ell+1}\,\frac{df_0}{dv}\,dv, \qquad \ell = 0, 2, \ldots, 2(\mathcal{N}-1). \tag{95}$$

A water bag model with $\mathcal{N}$ bags is equivalent to a continuous distribution function for moments up to $\ell_{\max} = 2(\mathcal{N}-1)$. Here we see that the equivalence in the fluid moment sense of a multiple water bag distribution and a continuous distribution makes the connection with a multifluid model more clear. Nevertheless equation (95) has the form of a Vandermonde system which becomes ill-conditionned for higher values of the number of bags $\mathcal{N}$ (for instance for $\mathcal{N} = 15$ and a cut-off in velocity space $v_{0\mathcal{N}} = 5v_{th}$, v_{th} being the thermal velocity, the matrix elements vary from 1 to 5^{28}!).

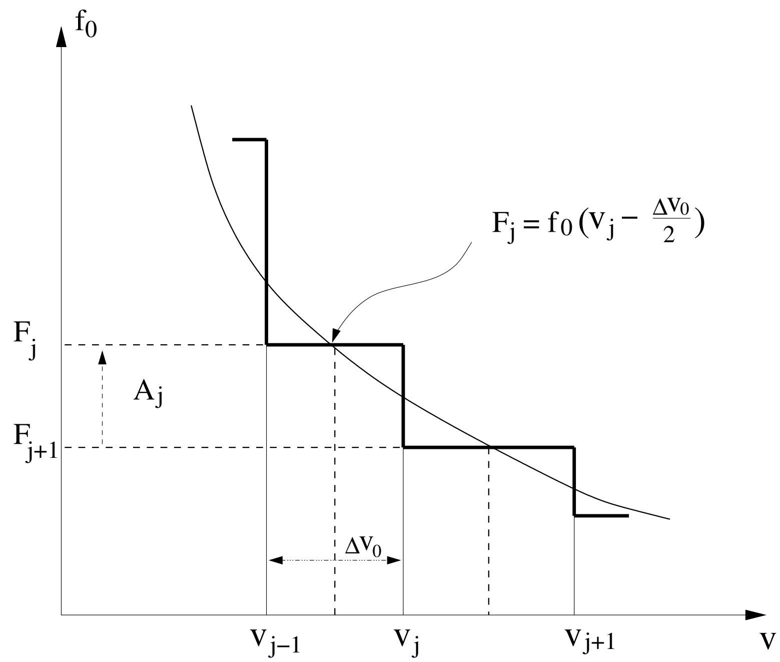


FIGURE 3. *Constructing the bags from a continuous distribution.*

A more convenient solution can be found for a regular sampling $v_{0j} = (j - \frac{1}{2})\Delta v_0$ and is explained in figure 3: we consider F_j at the middle of the interval $\Delta v_0 = \frac{2v_{0\mathcal{N}}}{2\mathcal{N}-1}$ and compute $F_j = f_0(v_{0j} - \frac{\Delta v_0}{2})$. From equation (95) the solution is straighforward

$$\mathcal{A}_j = f_0\left(v_{0j} - \frac{\Delta v_0}{2}\right) - f_0\left(v_{0j} + \frac{\Delta v_0}{2}\right) + \mathcal{O}(\Delta v_0^3). \tag{96}$$

In the following numerical experiments we assume a normalized Maxwellian distribution ($v_{th} = 1$) for f_0. Therefore the initial condition $v_{0j}^{\pm}$ is taken as

$$v_{0j}^{\pm} = \pm v_{0j}(1 + \eta\delta v(z)), \tag{97}$$

where η is real small number and δv is a periodic function in z (usually a cosine fonction).
Moreover it is well known that the Vlasov equation conserves many physical and mathematical quantities such that mass, kinetic entropy, total energy, every L^p-norm ($p \geq 0$) and more generaly any phase-space integral of $\beta(f)$ where β is a regular function. Obviously these conservation properties are retrieved with the water-bag model, by using the distribution function (6) in the definition of the considered quantities. For example the total energy, preserved in our water-bag model, is

$$\frac{1}{6}\sum_j \mathcal{A}_j \int dz\, \left({v_j^+}^3 - {v_j^-}^3\right) + \frac{1}{2}\sum_j \mathcal{A}_j \int dz\,(v_j^+ - v_j^-)\phi$$
$$+ \frac{1}{2}\sum_j \mathcal{A}_j \int dz\,(v_j^+ - v_j^-)|A_\perp|^2 dz + \frac{1}{2}\int dz\, \left(|\partial_z A_\perp|^2 + |\partial_t A_\perp|^2\right).$$

5.2. **Landau damping of Langmuir waves.** In this section we investigate the linear Landau damping of Langmuir waves corresponding to a damping of the wave without energy dissipation occuring by a phase mixing process of real frequencies [57, 9] which is reminiscent of the Van Kampen-Case [61, 24] treatment of electronic plasma oscillations. Initially all the bags are in phase. As time goes on the bags become more and more out of phase because each bag has its own phase velocity (determined by its own frequency, the real roots of the dispersion relation, and the wave number of excited mode) which differs from one bag to another one. This phase mixing between the bags is responsible for the linear Landau damping of the Langmuir waves. Further in time, there exists a recurrence time when all the bags are again in phase like at the begining, and thus the electrical waves recover their initial energy. In the asymptotic $v_\phi = \omega/k \gg v_{th}$ the dispersion relation for Vlasov-Poisson system with a Maxwellian distribution as the unperturbed part of the full distribution function gives for the frequency of the oscillation

$$\omega^2 = \omega_p^2 + \frac{3k_B T_e}{m_e}k^2$$

and for the damping rate

$$\gamma = -\sqrt{\pi}\omega_p \left(\frac{\omega_p}{kv_{th}}\right)^3 \exp\left(\frac{-\omega_p^2}{k^2 v_{th}^2}\right) \exp(-3/2).$$

The parameter setting is $L = 4\pi$, $v_{th} = 1$, $\mathcal{N} = 16$, $v_{max} = 6$ and $n_0 = 1$. The initial data are according to (97) with δv as a sine function. The oscillation frequency and the damping rate given by the numerical solution of the system (69)-(70) are respectively $\omega = 1.415$ and $\gamma = -0.153$, which is in agreement with the theoretical values $\omega = 1.4156$ and $\gamma = -0.1533$. Moreover the theoretical recurrence time $T_R = 2\pi/(k\Delta v)$ is equal to 32.46 which is in agreement with that observed on figure 4. In addition the relative error of variations of L^2-norm, kinetic entropy and mass or L^1-norm always stay less than 10^{-13}. The relative error on the total energy variation remains less than 10^{-8} for mesh discretization $\Delta x = 0.7862$, $\Delta v = 0.325$. The conservation properties of the discretized multi-water-bag model are better than those obtained by classical semi-Lagrangian kinetic schemes. For the same test case we obtain relative error variations smaller than 10^{-5} for the mesh discretization $\Delta x = 0.3925$, $\Delta v = 0.1875$ in [14] and relative error variations smaller than 10^{-5} for the mesh discretization $\Delta x = 0.3925$, $\Delta v = 0.25$ in [56]. In fact for semi-Lagrangian schemes the relative error variations of the conserved quantities increase when the distribution function is smoothed, i.e. when the size of the structures generated by the flow in the phase space becomes smaller than the phase space cell size. This phenomenon is less strong in the water-bag model because we only have to follow the contour dynamics while the phase space density between two neighbouring contours does nor need to be solved since the solution is obviously analytically known to be a constant.

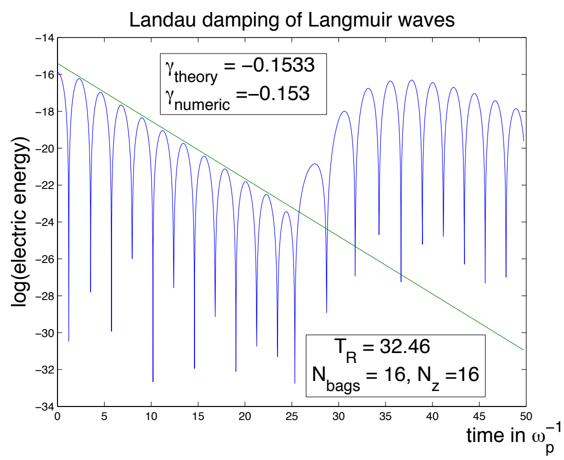


FIGURE 4. Evolution in time of the Logarithm of electric energy.

5.3. **Landau damping of ion acoustic waves.** If a wave has a slow enough phase velocity to match the ion thermal velocity, the ion landau damping phenomenon can occur. The dispersion relation for ion wave is

$$\frac{\omega}{k} = v_s = \left(\frac{Z_i k_B T_e + 3k_B T_i}{m_i}\right)^{1/2}.$$

If $T_e \leq T_i$ or $T_e \sim T_i$, the phase velocity lies in the region where the Maxwellian unperturbed part of distribution function has a negative slope. Consequently ion waves are heavily Landau-damped. On the other hand ion waves propagate without damping if $T_e \gg T_i$ since the phase velocity lies far in the tail of the ion velocity distribution. For a single ion species, for $k\lambda_D \ll 1$ (λ_D is the Debye length) the dispersion relation writes

$$Z'\left(\frac{\omega}{kv_{th}}\right) = \frac{2\,T_i}{Z_i T_e} = \frac{2\,\tau}{Z_i},$$

where $Z(\zeta) = \pi^{-1/2}\int_{-\infty}^{\infty} e^{-t^2}/(t-\zeta)dt$ stands for the plasma dispersion function. The numerical value of the parameters are $L = 4\pi$, $v_{th} = 1$, $\mathcal{N} = 16$, $v_{max} = 6$, $n_0 = 1$, $Z_i = 1$ and $\tau = 0.5$. The initial data are according to (97) with δv as a cosine function. The damping rate given by the numerical solution of the system of equations (69) and (71) is $\gamma = 0.288$ which is in good agreement with the theoretical value $\gamma = 0.290$. Moreover the theoretical recurrence time $T_R = 2\pi/(k\Delta v)$ is equal to 32.46 which is in agreement with that observed on figure 5. In addition the relative error variations for L^2-norm, kinetic entropy, total energy and mass or L^1-norm remains less than 10^{-12}.

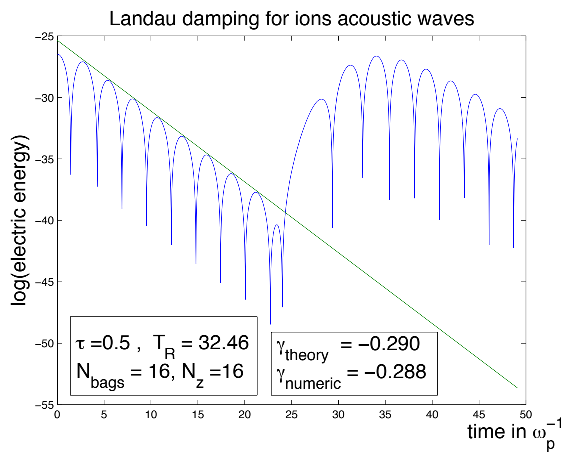


FIGURE 5. Evolution in time of the Logarithm of electric energy.

5.4. **Nonlinear Bohm-Gross frequency shift of a plasma wave.** In this section we take a plasma in a periodic box of length $L = 2\pi/k_0$ and we consider the initial conditions in the one bag case

$$v^{\pm}(t=0,z) = \pm v_0(1 + \varepsilon\cos(kx)),$$

where $k = \ell k_0$ (ℓ integer), and $\pm v_0$ are the unperturbed part of v^+ and v^-. Since all velocities are normalized to the thermal velocity, we have $v_0 = \sqrt{3}$ and $A = A_1 = (2\sqrt{3})^{-1}$. Therefore, the initial density $n(t=0,z) = A(v^+ - v^-)$ and mean velocity $u(t=0,z) = (v^+ + v^-)/2$ are written:

$$n(t=0,z) = 1 + \varepsilon\cos(kz) \quad \text{and} \quad u(t=0,z) = 0.$$

The system (69)-(70) with one bag ($\mathcal{N} = 1$) is simpler than the Vlasov-Poisson system governing a collisionless plasma and allows some analytical work ([8, 28]) in the weak field approximation ($\varepsilon \to 0$). In the linear case we obtain the Bohm Gross dispersion relation

$$\omega_k^2 = 1 + 3k^2. \tag{98}$$

In the Maxwellian case, this expression is valid for $k \to 0$, neglecting $\mathcal{O}(k^4)$-term, while in the linearized water bag model it is an exact result. Pushing now calculation up to third order in ε, using a multiple time scale perturbation method a new dispersion relation is obtained [8]

$$\omega_k' = \omega_k + \frac{\varepsilon^2}{16}\left(\frac{(1+\omega_{2k}^2)^2}{12\omega_k} + \frac{\omega_k}{3}(2 + 3\omega_{2k}^2) - \frac{2}{\omega_k\omega_{2k}^2}\right), \tag{99}$$

where ω_k is given by (98) and ω_{2k} is the corresponding formula for the mode $2k$.

Here we try to recover the non-linear frequency (99). The initial conditions for the parameters are $\varepsilon = 0.1$ and $k = k_0 = 0.6$ (i.e excitation of the first Fourier mode). Since $n(t,k_0)$ behaves like $\varepsilon/2\cos(\omega_{k_0}'t)$, we plot in Fig. 6 the difference $n(t,k_0) - \varepsilon/2\cos(\omega_{k_0}t)$ which must oscillate with an amplitude varying like $\varepsilon\sin((\omega_{k_0}' - \omega_k)t)/2$. For $\varepsilon = 0.1$ and $k = k_0 = 0.6$ the equation (99) gives $\omega_{k_0}' - \omega_k = 6.67\,10^{-3}$. Thus we obtain a straightline envelope with slope $3.357\,10^{-4}$ which is just the analytical value $\varepsilon/2(\omega_{k_0}' - \omega_{k_0})$, providing full support for the code.

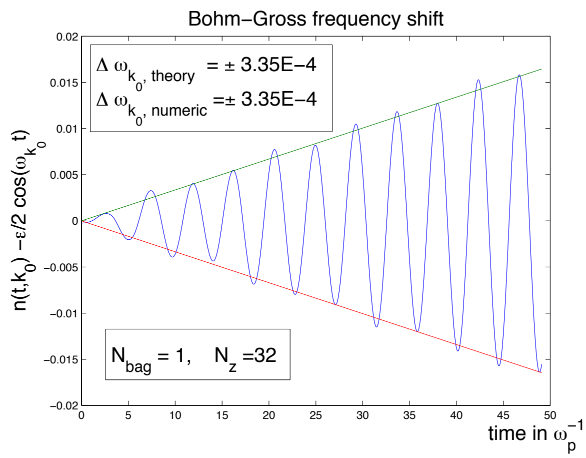


FIGURE 6. Evolution in time of $n(t, k_0) - \varepsilon/2\cos(\omega_{k_0} t)$.

5.5. **The Van Kampen modes.** The Van Kampen modes are the eigenmodes of the linearized Poisson-MWB system (69)-(70). If we linearize equations (69)-(70) for a periodic electronic plasma around an homogeneous (density n_0) equilibrium i.e. $v_j^{\pm}(t,z) = \pm v_{0j} + \delta v_j^{\pm}(t,z)$ with $|\delta v_j^{\pm}| \ll v_{0j}$, we then obtain the equations for the perturbation $\delta v_j^{\pm}(t,z)$

$$\partial_t v_j^{\pm} \pm v_{0j}\partial_z \delta v_j^{\pm} = -E_z, \quad \partial_z E_z = -\sum_{j=1}^{\mathcal{N}} \mathcal{A}_j(\delta v_j^{+} - \delta v_j^{-}). \tag{100}$$

After taking the Fourier transform of equations (100) and assuming that the time dependence of the Fourier mode $\mathcal{A}_j \delta v_{jk}^{\pm}(t)$ is of the form $\mathcal{A}_j \delta v_{jk}^{\pm}(t) = w_{jkn}^{\pm}\exp(-\omega_n t)$ we find the equation

$$kv_{0j}w_{jkn}^{\pm} + \frac{\mathcal{A}_j}{k}\sum_{i=1}^{\mathcal{N}}(w_{ikn}^{+} - w_{ikn}^{-}) = \omega_n w_{jkn}^{\pm}. \tag{101}$$

If we assume the normalization condition $\sum_{i=1}^{\mathcal{N}}(w_{ikn}^{+} - w_{ikn}^{-}) = 1$ which is equivalent to the dispersion relation (13) ($\epsilon(k,\omega) = 0$) we obtain from equation (101) the water-bag eigenmode $w_{jkn}^{\pm}\exp(-\omega_n t)$ where

$$w_{jkn}^{\pm} = \frac{1}{k}\frac{\mathcal{A}_j}{(\omega_n \mp kv_{0j})}. \tag{102}$$

The water-bag mode is very similar to the Van Kampen mode [61] (solution of the linearized Vlasov-Poisson system)

$$\chi_k(\omega, v) = -\frac{\partial_v f_0}{k}\text{p.v.}\frac{1}{\omega - kv} + \lambda(\omega)\delta(v - \omega/k),$$

where $-\partial_v f_0$ and $\mathcal{A}_j$ play the same part (see section 5.1) and $\lambda(\omega)$ is determined by the normalization condition $\int_{\mathbb{R}} \chi_k(\omega, v)dv = 1$. Let us notice that the Dirac distribution which is present in the Van Kampen mode desappears in the water-bag mode (102) because the phase velocity ω_n/k of the water-bag mode strictly lies between two consecutive bags v_{0j}. In fact the water-bag modes whose frequency spectrum is discrete and finite on the real axis appear as the discretization of the

Van Kampen modes whose frequency spectrum is dense on the real axis. The general solution of the system (100) is obtained by a linear combination of the water bag eigenmodes (102), i.e. $\mathcal{A}_j\delta v_{jk}^{\pm}(t) = \sum_n C_n w_{jkn}^{\pm}\exp(-\omega_n t)$ where the C_n's are determined by the initial condition. The summation over the index n which corresponds to the superposition of free oscillations is responsible for the Landau damping as already explained in 5.2. Here we want to excite a unique mode (k, ω_ℓ), i.e $C_n = \varepsilon\delta_{n\ell}$ where (k, ω_ℓ) satisfy the dispersion relation (13) ($\epsilon(k, \omega_\ell) = 0$). From equation (102) the initial condition is such that $\delta v_{jk}^{\pm}(t = 0) = (\varepsilon/k)/(\omega_\ell \mp kv_{0j})$, and the corresponding solution of the problem (100) is the traveling wave

$$\delta v_j^{\pm}(t, z) = \frac{\varepsilon}{k}\frac{1}{(\omega_\ell \pm kv_{0j})}\cos(kz - \omega_\ell t)$$

propagating at the phase velocity $v_{\varphi,\ell} = \omega_\ell/k$ and the associated density becomes $\delta n(t, z) = \sum_{i=1}^{\mathcal{N}}\mathcal{A}_j(\delta v_j^+ - \delta v_j^-) = \varepsilon\cos(kz - \omega_\ell t)$. Here we choose $k = 2.72$, $L_z = 2\pi n_k/k$ where $n_k = 13$. Solving the dispersion relation (13), we find $\omega_\ell = 1.68 \times 10^{-1}$. The phase velocity of the mode $v_{\varphi,\ell} = 6.19 \times 10^{-2}$ lies between the first and second contour and thus these two bags will be the most distorted ones. The other parameters are $v_{th} = \sqrt{1/511}$, $\mathcal{N} = 4$, $v_{max} = 0.22$, $L_z = 30.02$ and $\varepsilon = 10^{-3}$. The final time of the simulation is $T = 100\omega_p^{-1}$. Hereafter table 1 gives the L^∞-error between the exact solution and the numerical one, and the corresponding rate of convergence. Since we choose polynomials of degree two and a third-order Runge-Kutta scheme, the Runge-Kutta-discontinuous-Galerkin method using upwind numerical fluxes should converges with a rate $\min\{3, n + 1\} = 3$. From numerical convergence rates summarized in table 1, we can conclude that the

N_z	$\lVert v_{h,1}(T) - v_1(T)\rVert_{L^\infty}$	order	$\lVert n_h(T) - n(T)\rVert_{L^\infty}$	order
128	4.42×10^{-6}		1.77×10^{-5}	
256	5.89×10^{-7}	2.90	2.85×10^{-6}	2.66
512	7.48×10^{-8}	2.97	3.84×10^{-7}	2.90
1024	9.41×10^{-9}	2.99	4.87×10^{-8}	2.98

TABLE 1. L^∞-error and convergence rate.

scheme reproduces theoretical results with the right order of accurary.

5.6. **The stimulated Raman scattering instability.** The stimulated Raman scattering instability is a parametric instability involving three waves: the incident electromagnetic wave, here reffered to as the "pump" wave (k_0, ω_0) which drives two unstable waves; a scattered electromagnetic wave (k_s, ω_s); and an electron plasma wave (k_e, ω_e). The Raman instability occurs when the usual matching conditions hold, $\omega_0 = \omega_s + \omega_e$ and $k_0 = k_s + k_e$, with the dispersion relation for the electron plasma wave given by the (normalized) Bohm-Gross frequency $\omega^2 = 1 + 3k^2v_{th}^2$ and for the two electromagnetic waves by $\omega^2 = 1 + k^2$. The matching conditions can be satisfied only if $n_e/n_{crit} < 1/4$ where n_{crit} is the critical density above which electromagnetic radiation will not propagate. The periodic boundary conditions imply the selection of different wave numbers to obtain either forward $(\omega_s/k_s > 0)$ or backward $(\omega_s/k_s < 0)$ scattering. If we set $k_0 = rk_\ell$ where r is a rational number p/q, then the constraints of the problem leads to the bi-quadratic equation

$$((2r - 1 - 3v_{th}^2)^2 - 12v_{th}^2(r - 1)^2)k_e^4 + (-6v_{th}^2 - 4r^2 + 4r - 2)k_e^2 - 3 = 0.$$

Here we solve the system formed by the equations (69)-(70) and (72). We start with an initial homegeneous Maxwellian distribution with a thermal velocity $v_{th} = \sqrt{0.1/511}$. The cutoff in velocity space is $v_{\max} = 0.07$. The plasma is embedded in a periodic box of length $L = 10.75$. A right ($\nu = +1$) circularly polarized electromagnetic pump wave $(E^0_\perp, B^0_\perp, A^0_\perp)$ is initialized in a simulation box such that

$$E^0_{\perp\,1}(t=0,z) = E_0\cos(k_0 z), \quad E^0_{\perp\,2}(t=0,z) = \nu E_0 \sin(k_0 z). \tag{103}$$

The field amplitude E_0 is linked to the momentum amplitude p_{osc} (the so-called "quiver momentum") of the oscillatory motion of an electron in the wave plane by the normalized relation $p_{osc} = E_0/\omega_0$. In the present simulation we take $p_{osc} = 10^{-2}$. The corresponding magnetic components write

$$B^0_{\perp\,1}(t=0,z) = -\nu E_0 \frac{k_0}{\omega_0}\sin(k_0 z), \quad B^0_{\perp\,2}(t=0,z) = E_0 \frac{k_0}{\omega_0}\cos(k_0 z). \tag{104}$$

The corresponding initial condition for the transverse vector potential $A_\perp$, are then given by

$$A^0_{\perp\,1}(t=0,z) = \frac{E_0}{\omega_0}\sin(k_0 z), \quad A^0_{\perp\,2}(t=0,z) = -\nu\frac{E_0}{\omega_0}\cos(k_0 z). \tag{105}$$

We take similar expressions for the scattered wave $(E^s_\perp, B^s_\perp, A^s_\perp)$ with $a_s = 10^{-6}$. We set $r = 2/3$ and wave numbers are chosen such that $k_e/k_z = 6$, $k_0/k_z = 4$ and $k_s/k_z = -2$. The other parameters are $\omega_0 = 2.54$, $\omega_s = 1.54$, $\omega_e = 1.$, $k_0 = 2.33$, $k_s = -1.17$, $k_e = 3.5$, $n_0/n_{crit} = 1.55 \times 10^{-1}$, $\mathcal{N} = 6$, $N_z = 128$ and $\Delta t = 1.68 \times 10^{-2}$.

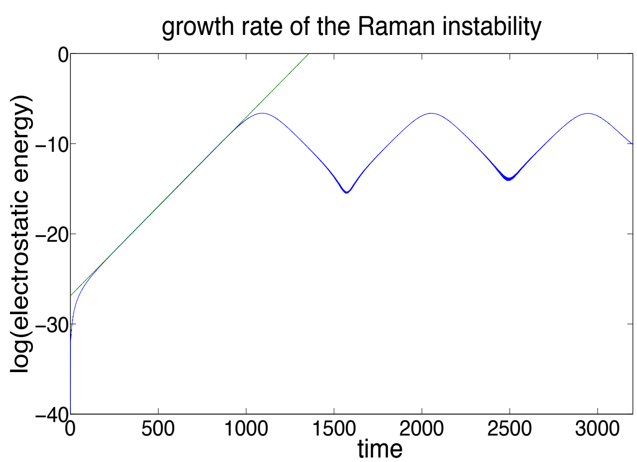


FIGURE 7. Growth rate of the stimulated Raman scattering instability.

At the first stage of the evolution the electric energy exhibit an exponential growth related to the SRS intability. The theoretical energy growth rate, deduced from linearized fluid equation [33] is

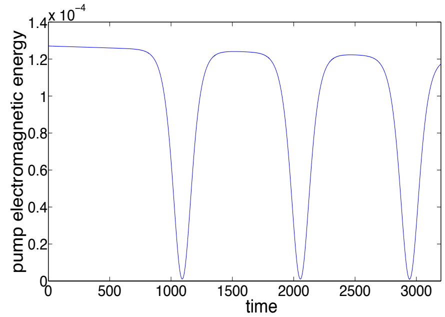
$$\gamma = \frac{k_e p_{\rm osc}}{2\sqrt{2\omega_e\omega_s}} = 9.97 \times 10^{-3}$$

and is found very close to the numerical value 9.90×10^{-3}, see Fig. 7. After the first stage of the SRS instability the time evolution of waves and particles energy in Fig. 8 exhibits an oscillatory behaviour in which energy is transfered back and forth between the pump, the scattered and plasma wave like a parametric 3-mode coupling.

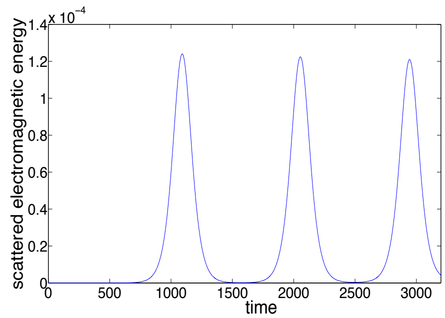
5.7. **The kinetic electron electrostatic nonlinear waves.** The KEEN (Kinetic Electron Electrostatic Nonlinear) waves are electrostatic acoustic-like modes of the one-dimensional Vlasov-Poisson system which propagate with a phase velocity around the thermal velocity and can be viewed as non-steady variants of the well-known Bernstein-Greene-Kruskal (BGK)[13] modes that describe invariant traveling electrostatic waves in plasmas. An explanation for the existence of these modes, which refers to Van Kampen-Case solution of the linearized Vlasov-Poisson system, were given by Holloway and Dorning [47]. The KEEN waves would be associated to the excitation of a Van Kampen mode around the phase velocity $v_\varphi = \omega/k \sim v_{th}$. We then obtain a dispersion diagram where the Bohm-Gross branch (corresponding to the undamped Landau pole) joins a balistic or acoustic branch [47, 50, 51](see Fig. 4 in [47]). An other way to explain the existence of such modes comes from the Landau solution of the linearized Vlasov-Poisson system. In fact from linear dispersion relation we can see that there exist an infinite number of poles beyond the Landau pole whose contributions are rapidly damped and which play a role only on very short time. If we can modify the initial distribution f_0 so as to flatten it around the phase velocity of the less undamped poles, then a structure can appear and propagates with phase velocity smaller than the Landau pole one. There must be a mechanism to excite such pole whereas Landau pole is naturally excited by the electronic density perturbation. This mechanism is linked to a ponderomotive force generated by the optic mixing of laser waves as it occurs in inertial fusion confinement. Therefore we are interested in the numerical solution of the system formed by the equations (69)-(70) and (72).

From dispersion relation (13) with $\mathcal{N}$ bags, for large wave length $k\lambda_D \ll 1$, we can see that the last pole $\omega_\mathcal{N}$ corresponds to the Landau pole, which is subjected to the Bohm-Gross relation $\omega_\mathcal{N}/\omega_p = 1 + 3k^2\lambda_D^2 + \mathcal{O}(k^4\lambda_D^4)$ whereas the $\mathcal{N} - 1$ other poles $\omega_{n<\mathcal{N}}$ are such that $\omega_n/\omega_p \sim k\lambda_D$, which corresponds to acoustic-like waves. These last acoustic-like water-bag modes can resonate with the electromagnetic branch to give a backward Raman scattering-like effect. The phase mixing can prevent these modes to develop and only Raman scattering (backward or forward) can resonate with the Bohm-Gross branch. Nevertheless, if we introduce a laser wave whose frequency and wave number are in accordance with one of those other acoustic-like water-bag pole, then this mode can propagate. In order to prevent resonance between the Bohm-Gross mode and the electromagnetic branch the condition $n_e/n_{crit} > 1/4$ must be satisfied. Let (k_0, ω_0) be the pump wave, and (k_s, ω_s) the scattered electromagnetic wave (with small amplitude) chosen such that one of the acoustic-like water-bag pole (k_ℓ, ω_ℓ) (with $\ell < \mathcal{N}$) comes in resonance (with $k_s < 0$), i.e. $\omega_0 = \omega_s + \omega_\ell$ and $k_0 = k_s + k_\ell$, then a unique acoustic-like water-bag pole can be excited. Moreover the electromagnetic waves and the water-bag mode must satisfy the dispersion relations $\omega^2 = 1 + k^2$ and $\epsilon(k_\ell, \omega_\ell) = 0$ respectively. If we set $k_0 = rk_\ell$ where r is a rational number p/q or a real number very close to a rational number, then the constraints of the problem lead to the second degree equation

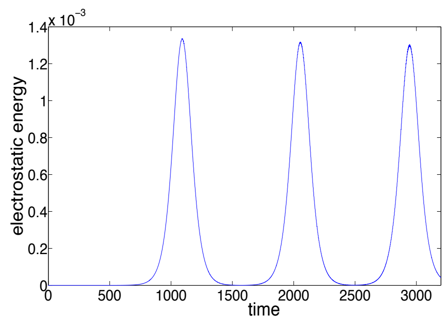
$$-r^2 + r + \frac{\omega_\ell^2 - k_\ell^2}{4k_\ell^2} - \frac{\omega_\ell^2}{k_\ell^2}\frac{1}{\omega_\ell^2 - k_\ell^2} = 0.$$



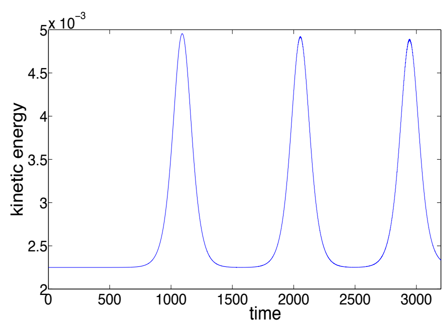
(a) pump electromagnetic wave energy



(b) scattered electromagnetic wave energy



(c) plasma wave energy



(d) particles kinetic energy

FIGURE 8. Time evolution of the energy of waves and particles.

The roots $r = 1/2$, $r > 1/2$, and $r < 1/2$ correspond respectively to $\omega_\ell = 0$, $\omega_\ell > 0$ and $\omega_\ell < 0$. We start with an initial homegeneous Maxwellian distribution with a thermal velocity $v_{th} = \sqrt{1/511} = 4.42 \times 10^{-2}$. The cutoff in velocity space is $v_{\max} = 0.22$. The plasma is embedded in a periodic box of length $L = 30.02$. A right circularly polarized electromagnetic pump and scattered wave are initialized in a simulation box with a quiver momentum $a_0 = p_{\rm osc} = E_0/\omega_0 = 10^{-2}$ and $a_s = 10^{-6}$. The structure of the initial pump $(E_\perp^0, B_\perp^0, A_\perp^0)$ and scattered $(E_\perp^s, B_\perp^s, A_\perp^s)$ wave are given by formula (103)-(105). We set $r = 7/13$ and wave numbers are chosen such that $k_\ell/k_z = 13$, $k_0/k_z = 7$ and $k_s/k_z = -6$. The other parameters are $\omega_0 = 1.77$, $\omega_s = 1.61$, $\omega_e = 1.68\times 10^{-1}$, $k_0 = 1.46$, $k_s = -1.26$, $k_e = 2.72$, $n_0/n_{crit} = 3.17 \times 10^{-1}$, $\mathcal{N} = 4$, $N_z = 256$ and $\Delta t = 2.34 \times 10^{-2}$. The phase velocity of the

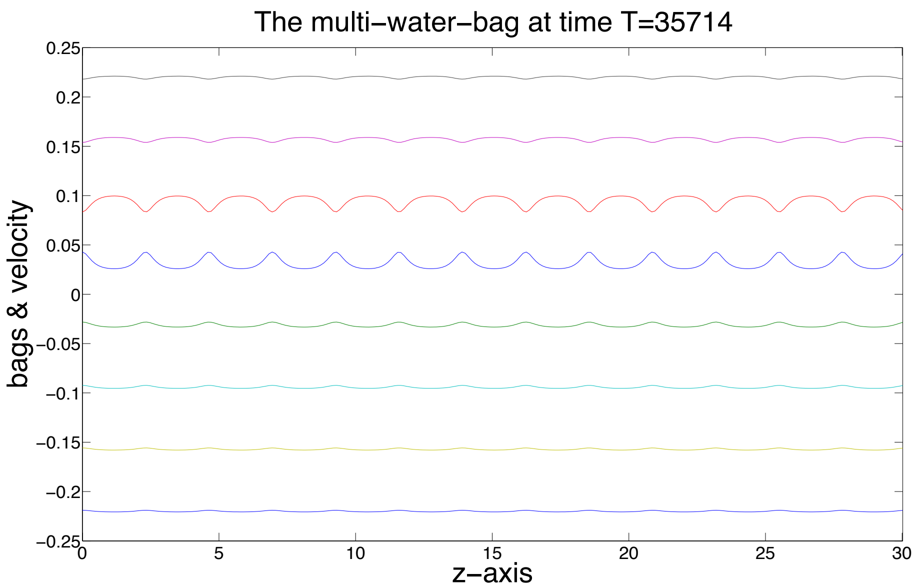


FIGURE 9. The muti-water-bag versus z-space at time $T = 35714$.

plasma mode is $v_{\varphi,\ell} = 6.19 \times 10^{-2} \sim 1.4 v_{th}$ whereas the theoretical velocity of the Bohm-Gross mode is $v_{\varphi,BG} = \sqrt{1 + 3k_\ell^2 v_{th}^2}/k_\ell = 3.75 \times 10^{-1}$ which is well beyond the cutoff velocity. In addition the relative error variations for L^2-norm, kinetic entropy, and mass or L^1-norm remain less than 10^{-10} whereas the relative error variation of the total energy is less than 4×10^{-2} at the final time $T = 3.5714 \times 10^4$. In Fig. 9, we observe the nonlinear stability in very long time of the low-frequency plasma mode that we have excited. In fact, in Fig. 9, we observe thirteen holes which correspond to as many vortices. This low-frequency nonlinear mode which moves with a velocity about the thermal velocity ($v_{\varphi,\ell} \sim 1.4 v_{th}$) is typically the wave that one observes in laser-plasma simulations using an electromagnetic Vlasovian description [2, 1, 34, 35, 16], the so-called KEEN mode. These modes can be viewed as a non-steady variant of the well-known Bernstein-Greene-Kruskal (BGK)[13] modes that describe invariant traveling electrostatic waves in plasmas. Therefore the multi-water-bag reveals itself as a model that can explain the formation of

KEEN waves and more generally supplies a scenario for the formation of coherent low-frequency structures which appear in laser-plasma interaction at nonlinear stage and persist in the long time dynamics such as electron acoustic-like waves (EAW). The ability of the multi-water-bag model to describe such waves is very promising and advanced research on this topic is under consideration.

6. **Conclusion.**

In this paper we have presented multi-water-bag models for collisionless kinetic equations. In fact the multi-water-bag model appears as a consequence of considering a special class of exact weak solutions of the Vlasov equation. On one hand, we have proved the existence of local classical solutions for the the multi-water-bag model to approximate one-dimensional Vlasov-type equations in three situations: the Poisson coupling, its quasi-neutral approximation and the electromagnetic coupling. On the other hand, we have proposed DG-type numerical approximations for these systems of equations. Moreover, we have shown the performance of this scheme based on the results of different test cases.

Let us notice that the water-bag model could appear somewhat limited when wave-breaking and extreme phase mixing occur like in the two stream instability case where vortices are shown to form. In this case the solution becomes multivalued and there are two ways to deal with this problem. The first way is to follow each branch of the solution and keep the Eulerian picture. The contours being still well defined in the phase space, even if the filamention phenomenon occurs, then it should be more convenient to adopt the Lagrangian description. From the numerical point of view every approach is a challenging difficult problem.

However, there remain some relevant hot topics in plasma physics such as gyrokinetic turbulence in magnetically confined thermonuclear fusion plasmas (ITER) [43] in which this model can be applied because in cylindrical geometry wave breaking or filamentation process are not dominant mechanisms [54, 55, 15]. Moreover this model has an advantage over classical gyrokinetic models due to the additional velocity variable reduction it affords, resulting in less expensive algorithms than ones followed from kinetic description. Moreover the multi-water-bag model reveals to be a useful and powerful tool to explain the formation of stable coherent low-frequency nonlinear structures as KEEN or electron acoustic-like waves which appear in laser-plasma interaction physics.

Appendix A. **A Gronwall lemma.**

Theorem A.1. *Let* v, f, $g \in \mathscr{C}([t_0,T),\mathbb{R}_+)$ *and* $\mathcal{F} \in \mathscr{C}(R_+^*,\mathbb{R}_+)$ *be a nondecreasing function with* $\mathcal{F}(v) > 0$. *If*

$$v(t) \le g(t) + \int_0^t \mathcal{F}(v(s))f(s)ds, \quad t_0 \le t < T,$$

then for $t_0 \le t < t_1$

$$v(t) \le \mathcal{G}^{-1}\left(\mathcal{G}(g(t)) + \int_0^t f(s)ds\right),$$

where

$$\mathcal{G}(x) = \int_1^x \frac{dy}{\mathcal{F}(y)},$$

and $t_1 \in (t_0,T)$ *is chosen such that*

$$\mathcal{G}(g(t)) + \int_0^t f(s)ds \in \mathrm{Dom}(\mathcal{G}^{-1}),$$

for all $t \in [t_0, t_1)$, where $\mathrm{Dom}(\mathcal{G}^{-1})$ *stands for the definition domain of* $\mathcal{G}^{-1}$.

Proof. Let us set $u(t) = g(t) + \int_0^t \mathcal{F}(v(s))f(s)ds$. Since $\mathcal{F}$ is not decreasing we get

$$\dot{u} = \dot{g} + \mathcal{F}(v)f \leq \dot{g} + \mathcal{F}(u)f. \tag{106}$$

Let us set now $\mathcal{G}(x) = \int_1^x \frac{dy}{\mathcal{F}(y)}$. Therefore $\mathcal{G}' = 1/\mathcal{F} \geq 0$ and thus $\mathcal{G}'$ is deacreasing and $\mathcal{G}$ is nondeacreasing. Besides $(\mathcal{G}^{-1})'(x) = [\mathcal{G}'(\mathcal{G}^{-1}(x))]^{-1} = \mathcal{F}(\mathcal{G}^{-1}(x)) \geq 0$ and thus $\mathcal{G}^{-1}$ is nondeacreasing. Using now the monotonicity of the functions $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}^{-1}$ and inequality (106) we get

$$\mathcal{G}'(u)\dot{u} \leq \mathcal{G}'(u)\dot{g} + f \leq \mathcal{G}'(g)\dot{g} + f. \tag{107}$$

An integration in time of equation (107) leads to

$$\mathcal{G}(u(t)) \leq \mathcal{G}(u(0)) - \mathcal{G}(g(0)) + \mathcal{G}(g(t)) + \int_0^t f(s)ds. \tag{108}$$

Since $u(0) = g(0)$ and $\mathcal{G}^{-1}$ nondecreasing, from equation (108) we obtain

$$v(t) \leq u(t) \leq \mathcal{G}^{-1}\left(\mathcal{G}(g(t)) + \int_0^t f(s)ds\right),$$

which ends the proof. □

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