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Asymptotic and spectral analysis of the gyrokinetic-waterbag integro-differential operator in toroidal geometry

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Achieving plasmas with good stability and confinement properties is a key research goal for magnetic fusion devices. The underlying equations are the Vlasov–Poisson and Vlasov–Maxwell (VPM) equations in three space variables, three velocity variables, and one time variable. Even in those somewhat academic cases where global equilibrium solutions are known, studying their stability requires the analysis of the spectral properties of the linearized operator, a daunting task. We have identified a model, for which not only equilibrium solutions can be constructed, but many of their stability properties are amenable to rigorous analysis. It uses a class of solution to the VPM equations (or to their gyrokinetic approximations) known as waterbag solutions which, in particular, are piecewise constant in phase-space. It also uses, not only the gyrokinetic approximation of fast cyclotronic motion around magnetic field lines, but also an asymptotic approximation regarding the magnetic-field-induced anisotropy: the spatial variation along the field lines is taken much slower than across them. Together, these assumptions result in a drastic reduction in the dimensionality of the linearized problem, which becomes a set of two nested one-dimensional problems: an integral equation in the poloidal variable, followed by a one-dimensional complex Schrödinger equation in the radial variable. We show here that the operator associated to the poloidal variable is meromorphic in the eigenparameter, the pulsation frequency. We also prove that, for all but a countable set of real pulsation frequencies, the operator is compact and thus behaves mostly as a finite-dimensional one. The numerical algorithms based on such ideas have been implemented in a companion paper [D. Coulette and N. Besse, "Numerical resolution of the global eigenvalue problem for gyrokinetic-waterbag model in toroidal geometry" (submitted)] and were found to be surprisingly close to those for the original gyrokinetic-Vlasov equations. The purpose of the present paper is to make these new ideas accessible to two readerships: applied mathematicians and plasma physicists. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4960742]

I. INTRODUCTION

A. Motivations and key issues

It is well known that plasmas confined by magnetic fields are often unstable. Indeed, the presence of density, temperature, velocity, and pressure gradients in the transverse direction of

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the magnetic confinement field constitutes free-energy sources, allowing small perturbations of the equilibrium state to grow exponentially, that is, the development of instabilities. This, in turn, leads to an important enlargement in the range of scales (particularly towards high wavenumber) present in the wave spectrum and to a significant increase in their amplitudes. In this wave-spectrum configuration, particles interacting with the waves have turbulent or chaotic dynamics. The study of such instabilities is important, in particular because they are the main cause of turbulent transport of heat and momentum.

Moreover, the thermal confinement of a magnetized fusion plasma is essentially determined by the turbulent heat conduction across the equilibrium magnetic field (in the transverse direction of the magnetic confinement field). Since the main energy loss in a controlled fusion device is of conductive nature, the energy confinement time has the same order of magnitude as the diffusion time a^2/χ_T, where χ_T is the thermal diffusivity and a is the transverse plasma size.

Consequently, the development of microscopic instabilities, through the generated microturbulence, can cause a dramatic reduction in the energy confinement time. The microturbulence stems from various instabilities (electrostatic, electromagnetic, fluid, . . .); furthermore, it may or not involve passing particle trajectories (open trajectories) or trapped ones (closed trajectories). The problem of accurately describing the status of all these possibilities for a given plasma is not yet completely solved. So far, many theoretical linear studies have been done on various microinstabilities to estimate their nonlinear saturation levels, the corresponding spectra, and the resulting transport across the equilibrium magnetic field.

Among all the microinstabilities, usually mentioned in investigating the stability of a magnetically confined plasma, those due to "flute-like" modes are particularly important when explaining anomalous energy transport in tokamak devices. The main property of these modes, justifying their name, is that $k_\parallel/k_\perp \ll 1$. Ion temperature gradient (ITG) modes are an example of such modes.

As far as turbulent diffusion is concerned, it is commonly observed that fluid simulations overestimate the turbulent diffusivity χ by roughly a factor two over the more accurate kinetic simulations.[29] Therefore, deciding which description to use may significantly impact the instability threshold and the growth rates and thus the predicted turbulent transport. The reason for this discrepancy is poorly understood, but wave-particle resonant processes (such as Landau damping) do certainly play an important part. Semi-empirical statistical approximations, known as closures, have been tried for this, but with little success so far (see, e.g., Ref. 92 and references therein).

The natural framework for studying turbulence and diffusion in the core of fusion plasmas is the six-dimensional kinetic collisionless models, such as the Vlasov–Poisson and Vlasov–Maxwell systems. Nevertheless, the presence of a very strong confining external magnetic field introduces a major simplification: to leading order, one obtains a helical cyclotronic motion (also called gyromotion) of the ions around the magnetic field lines. The radius of this helix is of order of the ion Larmor radius ρ_i, while the time frequency is of order of the ion cyclotron frequency Ω_i. Since the problem possesses an approximate symmetry (the ion gyromotion), a perturbation analysis can be applied to create an ignorable coordinate ζ (the gyro-angle, which parametrizes the ion helical motion) and thereby one has transformed the approximate symmetry (ion cyclotron motion) into an exact one (ion helical motion). Noether's theorem then provides a corresponding invariant, the adiabatic invariant μ (the magnetic moment), which together with the gyro-angle ζ constitutes a pair of conjugate variables. The gyrokinetic equation, parametrized by the magnetic moment μ, is obtained by averaging the Vlasov–Poisson or Vlasov–Maxwell along the gyro-angle ζ. The six-dimensional Vlasov equation has thus been reduced to a four-dimensional gyrokinetic equation, parametrized by the one-dimensional adiabatic invariant μ, where time frequencies larger than the ion cyclotron frequency Ω_i and wavelengths smaller than the ion Larmor radius ρ_i have dropped out.[38,30,58,17,16]

It is important that gyrokinetic simulations measure the discrepancy between the local distribution function and the Maxwellian distribution, used by most fluid closures. Note that, although more accurate, the gyrokinetic description of turbulent transport is much more demanding in computer resources than fluid simulations. This motivates us to revisit an alternative approach, based on the waterbag-like weak solution of kinetic equations.

The aim of this paper is to develop the linearized theory of collisionless kinetic flute-like waves, such as ionic instabilities (ITG modes), using an exact geometric reduction of the gyrokinetic-Vlasov equations. This reduction makes use of the "waterbag invariants," expressing, on the one hand, the conservation of the distribution function along phase-space characteristics and, on the other hand, the conservation of phase-space volume (Liouville's theorem). The waterbag model, which will be discussed in Secs. II-V, can be seen as a special class of weak solutions of the collisionless kinetic equations. It also constitutes a bridge between fluid and kinetic descriptions of a collisionless plasma, allowing to preserve the kinetic aspects of the problem (such as Landau damping and resonant wave-particle interactions), while possessing much lower complexity, namely that of a multi-fluid model. We believe that gyrokinetic-waterbag (or, simply, gyrowaterbag) models are very promising, insofar as they are amenable both to much analytical theory (thanks to their lower dimension) and to efficient numerical simulation.[6,5,7,9,8,10,2]

B. Presentation and explanation of the results

The main result is the design of an algorithm to construct eigenmode solutions for the linearized gyrokinetic-waterbag operator, here called the gyrowaterbag integro-differential operator (see Sec. III A). This construction relies on the asymptotic analysis of the eigenvalue problem (see Sec. IV) and the spectral analysis of integro-differential operators (see Sec. V) arising from the asymptotic analysis. We should note that we have been influenced by pioneering work of Refs. 22, 73, 72, 44, 21, 99, 90, 39, 28, and 70. Such work has provided us with complementary formalisms for the study in tokamaks, on the one hand, of two dimensional ideal magnetohydrodynamic modes and, on the other hand, of kinetic modes, the latter being based on a linearization of the Vlasov–Maxwell equations, *followed* by a gyrokinetic approximation (whereas, we use the gyrokinetic approximation first).

As usual in collisionless collective interactions, the particles described in the waterbag distribution function are coupled nonlinearly and self-consistently to the field — here the electrical potential ϕ — that they produce. In order to have a scalar problem we choose the electrical potential as the main unknown to write the eigenvalue problem in a closed form.

For describing microinstabilities and low-frequency waves in a toroidal plasma confinement system whose phase is approximately constant along a magnetic field line, but whose transverse vector is large, i.e., for $k_\parallel/k_\perp \ll 1$, we usually use the "ballooning formalism" first introduced in Ref. 22 to describe ideal magnetohydrodynamic ballooning instabilities driven by pressure gradients. For a detailed description and use of the ballooning formalism, see Sec. III C 2. Here we just extract what is necessary for an overview of our results.

Using the ballooning transform (see Sec. III C 2), the electrical potential fluctuation $\phi = \phi(t,\mathbf{r})$ with $\mathbf{r} \in \mathbb{R}^3$ reads

$$\phi(t,\mathbf{r}) = \sum_{(n,\ell)\in\mathbb{Z}^2} \sum_{\omega\in\mathscr{S}_n} \exp(-\mathrm{i}\omega t)\phi_{\omega n}(\theta + 2\pi\ell; q, \theta_{k0,T}) \exp\left(\mathrm{i}n\left[\varphi - q(\theta + 2\pi\ell - \theta_{k0,T})\right]\right) A_1(q), \quad (1)$$

where n is the toroidal wavenumber, q stands for a radial variable, φ and θ denote respectively the toroidal and poloidal angle (see Sec. II C for the description of the toroidal geometry). The constant angle $\theta_{k0,T}$ is called the ballooning angle, and the set $\mathscr{S}_n$ constitutes the point spectrum of our linear operator. The eikonal term $n\left[\varphi - q(\theta + 2\pi\ell - \theta_{k0,T})\right]$ in the decomposition (1) represents the fast variation of the solution in the radial variable q and poloidal angle θ. The constant ballooning angle $\theta_{k0,T}$ centers the solution poloidally. The poloidal envelope $\phi_{\omega n}(\theta; q, \theta_{k0,T})$, which depends parametrically on q and $\theta_{k0,T}$, gives the slow variation of the solution in the poloidal angle θ, while the radial envelope $A_1(q)$ determines the slow variation of the solution in the radial direction.

To obtain $\phi_{\omega n}$, $\theta_{k0,T}$, A_1, and $\mathscr{S}_n$, we roughly proceed as follows. For high toroidal wavenumbers n (flute-like modes), that is for $k_\parallel/k_\perp \ll 1$, we use a WKB-type analysis in a field-aligned coordinate system, to demonstrate that one can construct eigenmode solutions of the two-dimensional gyrowaterbag integro-differential operator. The corresponding equation in the (r,θ)-poloidal plane turns out not be a two-dimensional partial differential equation; instead, it reduces to two nested equations: a one-dimensional Fredholm-type integral equation and a one-dimensional non-self-adjoint Schrödinger equation (with complex potential).

Let us indicate some key ideas leading to such equations. First, by a suitable choice of the small parameter expressing the strong transverse/longitudinal anisotropy, we are able to decouple the radial and poloidal differential operators up to and including the second order.

To zeroth order, we obtain the slowly varying poloidal eigenmode envelope $\phi_{\omega n}(\theta) = \phi_{\omega n}(\theta; q, \theta_{k0})$ (see Sec. IV B and the Proposition 2), which satisfies the integral equation

$$\phi_{\omega n}(\theta) = \int_{-\infty}^{\infty} d\eta\, \mathbb{K}(\theta, \eta; \omega_0, q, \theta_{k0})\phi_{\omega n}(\eta), \tag{2}$$

where the kernel $\mathbb{K}(\theta, \eta; \omega_0, q, \theta_{k0})$ depends parametrically and nonlinearly on the local eigenfrequency ω_0, radial variable q and ballooning angle θ_{k0}. Let us point out that solving (2) amounts to solving for the mode geometry along the magnetic field lines locally in the radial variable. After solving (2), for each value of the couple of parameters (q, θ_{k0}), we obtain the local eigenfrequency $\omega_0 = \omega_0(q, \theta_{k0})$ that depends on the radial variable q and on the ballooning angle θ_{k0}. The zeroth-order solution contains also an arbitrary ballooning function $\theta_{k0}(q)$ which is determined from the first-order problem and found to be a constant that can be chosen to maximize the radial extension of the eigenmode. The study of the first-order problem also shows that the first-order correction to the eigenfrequency vanishes. Finally, from the second-order problem one obtains a linear Schrödinger equation for the determination of the radial eigenmode profile A_1 and the second-order global complex eigenfrequency $\omega \in \mathscr{S}_n$.

Some of the key results regarding the compactness of the integral operator in (2) are, of course, obtained by the detailed study of this operator. This requires, on the one hand, the use of standard results about compactness of weakly singular operators and integrability properties of the kernels involved in (2), and, on the other hand, a careful examination of the analytic continuation in complex eigenfrequency in connection with the boundary conditions used to integrate the zeroth-order equation (for statements of the theorems and detailed proofs, see Sec. V B).

Finally, let us note that our asymptotic approach allows us to prove the construction of normal modes whose radial extension is of order $n^{-\sigma}a$ with $\sigma > 1/2$ (a is the length scale of the small radius of the torus). Note that in the plasma physics literature, this exponent is frequently found to be exactly one half, rather than strictly greater to one half.[70,73,21,39] It would be of interest to find if this slight discrepancy is or not an artefact of using the waterbag model.

C. Advantages and drawbacks of the asymptotic approach

The asymptotic approach has one obvious advantage: solving one-dimensional integral equations has much lower complexity than tackling a two-dimensional partial differential equation problem to determine the whole spectrum, and furthermore is easily amenable to high parallelization. Also, of course, the underlying physics and mathematics emerge more clearly and are likely to lead to further theoretical work. The development of numerical schemes for solving the nested one-dimensional Fredholm-type equation is beyond the scope of the present paper. Such computations are presented in a companion paper for the quasilinear gyrowaterbag initial-value problem,[24] where it is shown that the asymptotic reduction to nested one-dimensional problems is very faithful, even when the toroidal wavenumber is only moderately large. Moreover, standard numerical results, obtained with full gyrokinetic-Vlasov codes without waterbag modeling,[29] show also fair agreement with our results.[24]

At the moment, there is no rigorous asymptotic theory for the high-toroidal-wavenumber expansion. Actually writing the equations beyond the second order is quite a challenge, but this is only a mathematical issue: beyond second order one loses the decoupling into nested one-dimensional problems and thus one ceases to gain in numerical complexity over the original problem. Still, if it turns out that one needs to determine modes with radial extension comparable to the small radius of the torus, then one cannot use our asymptotic theory and solution of the two-dimensional gyrokinetic-waterbag model becomes unavoidable.

D. Organisation of the paper

In Secs. II and II A, we recall the gyrokinetic framework and the gyrokinetic-Vlasov equation. In Sec. II B, we explain the waterbag reduction concept, beginning with a simple one-dimensional Vlasov model and then apply the waterbag reduction concept to the gyrokinetic-Vlasov equation to obtain the gyrokinetic-waterbag model. In Secs. II C and II D, we describe the magnetic field line geometry and the different scales of the problem. In Secs. III A, III B, and III C, we introduce the linearization of the gyrowaterbag model, explain how to analytically solve for the steady equilibrium state, and how to use field-aligned coordinates and the ballooning-eikonal representation to recast the system for the perturbations in a form suitable for its subsequent asymptotic analysis. In Secs. IV and IV F, we perform the asymptotic analysis of the linearized gyrowaterbag equation and describe an efficient algorithm for constructing eigenmode solutions. In Sec. V, we perform the spectral analysis of the linear operators that arise from the eigenvalue problem; more precisely the one-dimensional non-selfadjoint Schrödinger-type operator and the one-dimensional nested Fredholm-type integral operator with a nonlinear dependency on the eigenparameter. In Appendix A we give a glossary of the main notation of the paper. In Appendix B we present a rigorous derivation of the gyrokinetic-waterbag equations. In Appendix C we present the linearization of the gyrokinetic-waterbag equations in detail. This uses some approximations related to the toroidal geometry, given in Appendix D.

The length of the present paper, to some extent, reflects our desire to make the material accessible to both the community of applied mathematicians (not necessarily involved in plasmas) and to that of theoretical and numerical plasma physicists.

II. THE GYROKINETIC FRAMEWORK

A. The gyrokinetic-vlasov equation

Predicting turbulent transport in collisionless fusion plasmas requires solving the gyrokinetic-Vlasov equation for all species coupled to the Darwin or magnetostatic equations (low-frequency approximations of Maxwell equations in the asymptotic limit of infinite speed of light[11]). This gyrokinetic approach has been widely used in recent years to study low-frequency micro-instabilities in a magnetically confined plasma, which are known to exhibit a wide range of spatial and temporal scales. Within the gyrokinetic Hamiltonian formalism,[38,30,58,17,16] the Vlasov equation expresses the fact that the ion gyrocenter distribution function $f = f(t, \mathbf{r}, v_\parallel, \mu)$ is constant along gyrocenter characteristic curves in gyrocenter phase-space $(t, \mathbf{r}, v_\parallel, \mu) \in]0, T] \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^+$,

$$D_t f = \partial_t f + \mathbf{F_r} \cdot \nabla_\mathbf{r} f + \mathrm{F}_{v_\parallel} \partial_{v_\parallel} f = 0, \tag{3}$$

where the force vector-field $\mathbf{F} = (\mathbf{F_r}, \mathrm{F}_{v_\parallel})$ reads

$$\mathbf{F_r} = \frac{\mathbf{b}}{q_i B_\parallel^*} \times \nabla_\mathbf{r} \mathcal{H} + \frac{1}{m_i} \frac{\mathbf{B}^*}{B_\parallel^*} \partial_{v_\parallel} \mathcal{H}, \qquad \mathrm{F}_{v_\parallel} = -\frac{1}{m_i} \frac{\mathbf{B}^*}{B_\parallel^*} \cdot \nabla_\mathbf{r} \mathcal{H},$$

with the definitions

$$\mathcal{H} = \frac{1}{2} m_i v_\parallel^2 + \mu B + q_i \mathcal{J}_\mu \phi, \qquad \mathbf{B}^* = \mathbf{B} + \frac{m_i v_\parallel}{q_i} \nabla \times \mathbf{b}, \qquad B_\parallel^* = \mathbf{B}^* \cdot \mathbf{b}.$$

In the previous equations $\mathbf{B} = \mathbf{B}(t, \mathbf{r})$ denotes the magnetic field with B its Euclidean norm, while $\mathbf{b} := \mathbf{B}/B$ stands for the unit vector tangent to the magnetic field line. The magnetic moment $\mu := m_i v_\perp^2/(2B)$ is the first adiabatic invariant of the ion gyrocenter; m_i and $q_i := Z_i e$ are respectively the mass and charge of ions with $e > 0$ being the electron Coulomb charge. Finally, the integral operator $\mathcal{J}_\mu$ stands for the gyroaverage operator defined by

$$\mathcal{J}_\mu f(\mathbf{r}) = \frac{1}{2\pi} \int_0^{2\pi} d\zeta\, f(\mathbf{r} + \boldsymbol{\rho}(\zeta)), \tag{4}$$

where ζ is the gyroangle. The gyroradius vector $\boldsymbol{\rho}$ is given by

$$\boldsymbol{\rho}(\zeta) = \sqrt{\frac{2\mu}{q_i\Omega_i}}\hat{\mathbf{a}}(\zeta) = \frac{v_\perp}{\Omega_i}\hat{\mathbf{a}}(\zeta),$$

where $\hat{\mathbf{a}}(\zeta) = \hat{\mathbf{x}}\cos\zeta - \hat{\mathbf{y}}\sin\zeta$ is defined in terms of the fixed local unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \mathbf{b} = \hat{\mathbf{x}} \times \hat{\mathbf{y}})$. The gyrogauge invariance involves an arbitrary rotation of the perpendicular unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ around the parallel unit vector $\mathbf{b}$. Using Fourier transforms the gyroaverage operator reads

$$\begin{aligned}
\mathcal{J}_\mu f(\mathbf{r}) &= \int_{\mathbb{R}^3} d\mathbf{k}\hat{f}(\mathbf{k})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\frac{1}{2\pi}\int_0^{2\pi} d\zeta \exp(\mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho}) \\
&= \int_{\mathbb{R}^3} d\mathbf{k}\hat{f}(\mathbf{k})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\frac{1}{2\pi}\int_0^{2\pi} d\zeta \exp\left(\mathrm{i}k_\perp\frac{v_\perp}{\Omega_i}\cos(\zeta+\gamma)\right) \\
&= \int_{\mathbb{R}^3} d\mathbf{k}\hat{f}(\mathbf{k})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})J_0\left(\frac{k_\perp v_\perp}{\Omega_i}\right),
\end{aligned}$$

where $k_\perp^2 := k_{\hat{\mathbf{x}}}^2 + k_{\hat{\mathbf{y}}}^2$ and $\tan(\gamma) := k_{\hat{\mathbf{y}}}/k_{\hat{\mathbf{x}}}$. As usual, $\Omega_i := q_iB/m_i$ is the ion cyclotron frequency and J_0 is the Bessel function of the first kind of zeroth order.

The gyrokinetic-Vlasov equation (3), which describes the dynamics of ions gyrocenter, is coupled to an adiabatic electron response via the quasi-neutrality condition

$$-\nabla_\perp\cdot\left(\frac{n_{i0}}{B\Omega_i}\nabla_\perp\phi\right) + \frac{e\tau n_{i0}}{T_{i0}}(\phi - \lambda\langle\phi\rangle_\parallel) = 2\pi\frac{\Omega_i}{q_i}\int_{\mathbb{R}} dv_\parallel \int_{\mathbb{R}^+} d\mu\mathcal{J}_\mu f(t,\mathbf{r},v_\parallel,\mu) - n_{i0}, \tag{5}$$

which determines self-consistently the electrical potential ϕ from ion gyrocenter distribution function f. In quasi-neutrality equation (5) we set $\tau = T_{i0}/T_{e0}$ and $\lambda \in \{0,1\}$; the quantity $\langle\phi\rangle_\parallel$ denotes the average of the electrical potential ϕ over a magnetic field line (or surface for *irrational magnetic flux surface*).

Since the magnetic moment μ is not an independent variable but a parameter or a label related to an (adiabatic) invariant, we can consider the plasma as a superposition of a (possibly uncountable) collection of bunches of particles having the same initial magnetic moment μ. This standard approach is equivalent, mathematically, to considering solutions of the Vlasov equation (3), written as

$$f(t,\mathbf{r},v_\parallel,\mu) = \int_M f_\nu(t,\mathbf{r},v_\parallel)\delta_\nu(\mu)\mathfrak{m}(d\nu).$$

Here $\delta_\nu(\mu)$ is the Dirac mass, ν is a parameter belonging to some probability space M (presently, $M = \mathbb{R}^+$), $\mathfrak{m}$ is a probability measure on that space and f_ν are smooth functions, which still satisfy the Vlasov equation (3) with $\mu = \nu$. A particular useful instance of this, which is quite central in our approach, happens when μ is a discrete variable, taking finitely many or enumerably many values, labelled by an index ℓ, so that $\mathfrak{m}(d\nu) = \sum_\ell \varpi_\ell\delta(\nu - \mu_\ell)$, where ϖ_ℓ are positive constants. As a consequence the distribution function f can be recast as

$$f(t,\mathbf{r},v_\parallel,\mu) = \sum_\ell \varpi_\ell f_{\mu_\ell}(t,\mathbf{r},v_\parallel)\delta(\mu - \mu_\ell), \tag{6}$$

where the function $f_{\mu_\ell}(t,\mathbf{r},v_\parallel)$ satisfies the Vlasov equation (3) (with $\mu = \mu_\ell$), for all values of the index ℓ.

Remark 1. Let us note that the gyrokinetic-Vlasov equation (3) satisfies the Liouville theorem

$$\frac{d}{dt}\int_{\Omega(t)} B_\parallel^* d\mathbf{r}dv_\parallel = 0 \quad\Longleftrightarrow\quad \partial_t(B_\parallel^*) + \nabla_\mathbf{r}\cdot(B_\parallel^*\mathbf{F}_\mathbf{r}) + \partial_{v_\parallel}(B_\parallel^*\mathrm{F}_{v_\parallel}) = 0,$$

where $\Omega(t)$ is the image of any bounded phase-space volume element $\Omega(0)$ from the Lagrangian flow induced by the force field $\mathbf{F}$. *The Liouville theorem allows to recover the conservative form of the Vlasov equation (3), i.e.,*

$$\partial_t(B_\parallel^* f) + \nabla_\mathbf{r}\cdot(B_\parallel^*\mathbf{F}_\mathbf{r} f) + \partial_{v_\parallel}(B_\parallel^*\mathrm{F}_{v_\parallel} f) = 0. \tag{7}$$

The Liouville theorem is also the key ingredient to obtain the conservation laws associated to the gyrokinetic system (3) and (5), such as the conservation of mass, of energy, of the Boltzmann entropy, of the Casimir functionals $\Theta(f)$ (with $\Theta : \mathbb{R}^+ \to \mathbb{R}$, smooth functions), and of the L^p-norms $(1 \leq p \leq \infty)$ of f.

B. The gyrokinetic-waterbag model

In this section we present the derivation of the gyrowaterbag model by using the waterbag reduction concept that we apply first to a simple one-dimensional (1D) Vlasov equation. This highlights the concept without burdening the reader with the full complexity of the gyrokinetic-Vlasov multi-dimensional equation.

1. The waterbag reduction concept in 1D

Let us consider a 1D periodic (in x-space) collisionless plasma (with a 2D phase-space (x,v)) described by the Vlasov equation

$$\partial_t f + v\partial_x f + F\partial_v f = 0, \tag{8}$$

with $f = f(t,x,v)$. The force vector-field $F = F(t,x,v)$ is taken divergence-free: $\nabla_{x,v} \cdot F = 0$, and does not need to be specified here. At the initial time, the situation is as depicted in the left panel of Fig. 1. Introducing the bag heights $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$, as shown in the right panel, the initial distribution function reads (with $\mathcal{N} = 3$)

$$f(0,x,v) = \sum_{j=1}^{\mathcal{N}} \mathcal{A}_j \left(\Upsilon(v_j^+(0,x) - v) - \Upsilon(v_j^-(0,x) - v) \right). \tag{9}$$

Here v_j^+ and v_j^- denote contours or curves in phase-space (with $j = 1, \ldots, \mathcal{N}$) and Υ is the Heaviside unit step function.

The Liouville theorem expresses phase-space measure conservation, namely

$$\frac{d}{dt}\int_{\Omega(t)} dv dx = 0.$$

Here $\Omega(t)$ is the image of any bounded phase-space volume element $\Omega(0)$ from the Lagrangian flow induced by the force field F. This requires v_j^+ and v_j^- to remain smooth and not to cross (single-valuedness is not mandatory); as a consequence, the area between the contours v_j^+ and v_j^- is conserved and equal to a fixed initial constant. Moreover, since the advective form of the Vlasov equation (8) expresses the constancy distribution function along the characteristic curves, the bag heights $\mathcal{A}_j$ are invariants (constant) and the structure of the waterbag distribution function (9) is

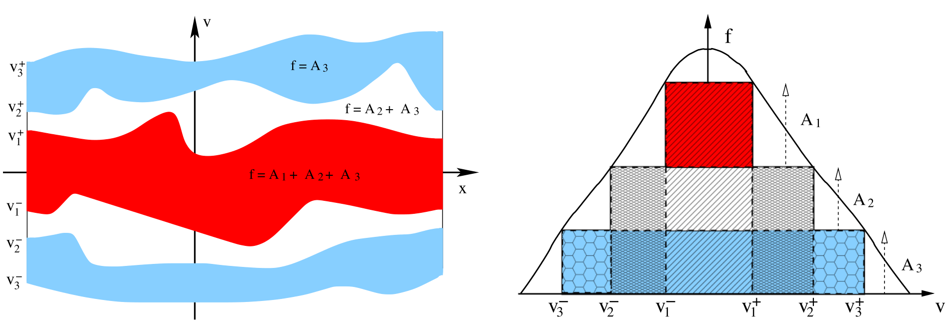

FIG. 1. *The waterbag reduction concept*: phase-space plot for a three-bag waterbag model (left panel) and corresponding waterbag distribution function (right panel); from a continuous distribution function f (right panel), we can obtain the waterbag distribution function (its velocity profile, right panel) with three bags by using a Lebesgue subdivision; in the right panel we observe the waterbag invariants (colored horizontal slices) and Liouville invariants (hatched vertical slices).

also preserved in time so that we obtain

$$f(t,x,v) = \sum_{j=1}^{N} \mathcal{A}_j \left(\Upsilon(v_j^+(t,x) - v) - \Upsilon(v_j^-(t,x) - v) \right). \tag{10}$$

Therefore the problem is entirely described by the constants $\mathcal{A}_j$ and the functions v_j^+ and v_j^- for the evolution equations are obtained as follows: observing that a particle on the contour v_j^+ (or v_j^-) remains on this contour, the equations for v_j^+ and v_j^- are

$$D_t v^\pm(t,x) = \partial_t v^\pm(t,x) + \left(v^\pm \partial_x v^\pm\right)(t,x) = F(t,x,v^\pm). \tag{11}$$

This equation can also be obtained directly by substituting the distribution function (10) into the Vlasov equation (8), in the sense of distribution theory.

Since a hydrodynamic description with several fluids, labelled by the index j, involves n_j, u_j, and p_j (respectively, density, average velocity, and pressure of the fluid j) we can predict the possibility of casting the waterbag model into the hydrodynamic frame with, in addition, an automatically provided equation of state. Indeed, let us define for each bag or fluid j, the density n_j, average velocity u_j, and pressure p_j such as $n_j = \mathcal{A}_j(v_j^+ - v_j^-)$, $u_j = (v_j^+ + v_j^-)/2$, and $p_j = n_j^3/(12\mathcal{A}_j^2)$. By adding and subtracting contour equations (11), for each bag or fluid j we recover the conservative form of the continuity (12) and Euler (13) equations (isentropic gas dynamics equations with $\gamma = 3$), namely

$$\partial_t n_j + \partial_x(n_j u_j) = 0, \tag{12}$$

$$\partial_t(n_j u_j) + \partial_x \left(n_j u_j^2 + p_j\right) = n_j F. \tag{13}$$

The geometric interpretation of the continuity and Euler equations (12) and (13) is that the shape (defined by the boundaries v_j^+ and v_j^-) of the bag j deforms, while its volume

$$\int n_j \, dx$$

is conserved in time. This is what we call the waterbag invariant. Obviously, we observe that we have reduced the kinetic Vlasov equation (8) to a multi-fluid hydrodynamic system (12) and (13). This is what we call the waterbag reduction concept, which is an exact reduction (we pass from a N-dimensional problem to a $(N-1)$-dimensional problem; here $N = 2$) based on Liouville invariants. Finally, another right and short way to see the waterbag reduction concept is just to consider a foliation of the phase-space by level lines, and solve the dynamics of the level lines.

The idea of using the many fluid structures to approximate collisionless kinetic equations seems to date from the sixties. In order to work with low-dimensional models for performing accurate numerical simulations with a tractable amount of data, physicists introduced first the waterbag model in plasma physics[27,3,4] and astrophysics.[67] Next mathematicians use the representation of many fluid structures (using Dirac and Heaviside distribution functions) to study rigorously the quasineutral limit of the Vlasov-Poisson equation[55,56] or to derive formal relations between the Vlasov equation and the semi-classical limit of the nonlinear Schrödinger equation.[106] In these works the idea is to use the nice properties of some fluid models such as hyperbolicity or convexity. The reciprocal idea to use a kinetic formulation of fluid equations to get a mathematical breakthrough in the well-posedness of nonlinear systems of conservation laws and to design new accurate numerical schemes dates from the eighties with a series of works.[12–15,43,74,75,81,82] In these works the waterbag or Heaviside function plays a crucial role. The idea is to take advantage of the linear structure of the kinetic equation (the so-called free-streaming equation) which is obtained from a lifting of the nonlinear conservation laws by using an extra variable, i.e., by increasing the dimension of the space.

2. The gyrokinetic-waterbag equations

Let us now assume that we deal with the discrete decomposition (6), where the number of values of the parameter μ is finite, say $\mathcal{M}$. Now, for every magnetic moment μ, we shall consider two three-dimensional foliations, denoted $\{v^{\pm}_{\mu b}(t,\mathbf{r})\}_b$, of the four-dimensional phase-space $(\mathbf{r}, v_\parallel)$, which may be viewed as families of three-dimensional smooth functions $\{v^{\pm}_{\mu b}(t,\mathbf{r})\}_b$, labelled by the one-dimensional index b, belonging to the set $[1,\ldots,\mathcal{N}]$. For every adiabatic invariant μ, we specify $2\mathcal{N}$ non-closed single-valued smooth contours $\{v^{\pm}_{\mu b}(t,\mathbf{r})\}_{b\le\mathcal{N}}$ of the $(\mathbf{r}, v_\parallel)$-phase-space ordered such that $\ldots < v^{-}_{\mu\, b+1} < v^{-}_{\mu b} < \cdots \le 0 \le \cdots < v^{+}_{\mu b} < v^{+}_{\mu\, b+1} < \cdots$, and some strictly positive real numbers $\{\mathcal{A}_{\mu b}\}_{b\le\mathcal{N}}$ that we call bag heights. In other words, for a fixed value of the parameter μ, the "plus" and "minus" branches $v^{\pm}_{\mu\, b}$ are monotonic with respect to the variable index b. For every value of the parameter μ, we then construct the distribution function $f_\mu(t,\mathbf{r},v_\parallel)$ such that

$$f_\mu(t,\mathbf{r},v_\parallel) = \sum_{b=1}^{\mathcal{N}} \mathcal{A}_{\mu b}\left(\Upsilon\left(v^{+}_{\mu b}(t,\mathbf{r}) - v_\parallel\right) - \Upsilon\left(v^{-}_{\mu b}(t,\mathbf{r}) - v_\parallel\right)\right). \quad (14)$$

As long as the contours are smooth, single-valued, and do not cross, the function (14) is an exact weak solution of the gyrokinetic-Vlasov equation (3) in the sense of distribution theory, if and only if the following gyrowaterbag equations in advective form are satisfied:

$$B^{*}_{\parallel}(v^{\pm}_{\mu b})\partial_t v^{\pm}_{\mu b} + \left(\frac{1}{q_i}\mathbf{b}\times\nabla\Phi + v^{\pm}_{\mu b}\mathbf{B}^{*}(v^{\pm}_{\mu b})\right)\cdot\nabla v^{\pm}_{\mu b} + \frac{1}{m_i}\mathbf{B}^{*}(v^{\pm}_{\mu b})\cdot\nabla\Phi = 0, \quad \forall(\mu,b).$$

After some algebra, the previous advective form of the gyrowaterbag equations can be written in conservative form as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left[\mathbf{B}+\mathbf{B}^{*}(v^{\pm}_{\mu b})\right]\cdot\mathbf{b}v^{\pm}_{\mu b}\right) + \nabla\cdot\left(\frac{1}{m_i q_i}\left(q_i\mathbf{A} + m_i v^{\pm}_{\mu b}\mathbf{b}\right)\times\nabla\mathcal{H}(v^{\pm}_{\mu b})\right) = 0, \quad \forall(\mu,b), \quad (15)$$

with the definitions

$$\mathbf{B} = \nabla\times\mathbf{A}, \qquad \mathbf{B}^{*}(v^{\pm}_{\mu b}) = \mathbf{B} + (m_i/q_i)v^{\pm}_{\mu b}\nabla\times\mathbf{b}, \qquad B^{*}_{\parallel}(v^{\pm}_{\mu b}) = \mathbf{B}^{*}(v^{\pm}_{\mu b})\cdot\mathbf{b},$$
$$\mathcal{H}(v^{\pm}_{\mu b}) = m_i v^{\pm 2}_{\mu b}/2 + \Phi, \qquad \Phi = q_i\mathcal{J}_\mu\phi + \mu B.$$

The quasi-neutrality coupling (5) can be rewritten as

$$-\nabla_\perp\cdot\left(\frac{n_{i0}}{B\Omega_i}\nabla_\perp\phi\right) + \frac{e\tau n_{i0}}{k_B T_{i0}}(\phi - \lambda\langle\phi\rangle_\parallel) = 2\pi\frac{\Omega_i}{q_i}\sum_{\ell=1}^{\mathcal{M}}\sum_{b=1}^{\mathcal{N}}\mathcal{A}_{\mu_\ell b}\mathcal{J}_{\mu_\ell}(v^{+}_{\mu_\ell b} - v^{-}_{\mu_\ell b}) - n_{i0}. \quad (16)$$

Remark 2. Since the force vector-field $\mathbf{F} = (\mathbf{F}_{\mathbf{r}}, \mathrm{F}_{v_\parallel})$ *is not divergence-free* ($\nabla_{\mathbf{r},v_\parallel}\cdot\mathbf{F}\neq 0$)*, we have*

$$\frac{d}{dt}\int_{\Omega(t)} d\mathbf{r}dv_\parallel \neq 0,$$

and thus the Liouville theorem is not satisfied in the variables $(\mathbf{r}, v_\parallel)$*. Therefore, we should not a priori use the variables* $(\mathbf{r}, v_\parallel)$ *and the waterbag distribution function (14) to apply the waterbag reduction concept, which leads to Equations (15). Introducing the Jacobian*

$$J(\mathbf{r},v_\parallel) = B^{*}_{\parallel}(\mathbf{r},v_\parallel) = \mathbf{B}\cdot\mathbf{b} + (m_i/q_i)v_\parallel\mathbf{b}\cdot\nabla\times\mathbf{b},$$

we observe that the Liouville theorem for the gyrokinetic-Vlasov equation (3) reads

$$\frac{d}{dt}\int_{\Omega(t)} J(\mathbf{r},v_\parallel)d\mathbf{r}dv_\parallel = 0. \quad (17)$$

The appearance of the Jacobian J *in (17) expresses that the variables* $(\mathbf{r}, v_\parallel)$ *are not canonical. Nevertheless, we can introduce the new "velocity" variable* $\xi_\parallel$*, which is defined as the primitive of* J *with respect to* $v_\parallel$*, i.e.,*

$$\xi_\parallel = \int^{v_\parallel} dv_\parallel J(\mathbf{r},v_\parallel).$$

The variables $(\mathbf{r},\xi_\parallel)$ *are now well suited to applying rigorously the waterbag reduction concept (by introducing the contours* $\xi^{\pm}_{\mu b}(t,\mathbf{r})$ *in the* $(\mathbf{r},\xi_\parallel)$*-phase-space). Indeed, we have the Liouville theorem*

$$\frac{d}{dt}\int_{\Omega(t)} d\mathbf{r}d\xi_\parallel = 0.$$

In Appendix B, we use the variables $(\mathbf{r}, \xi_\parallel)$ to rewrite the gyrokinetic-Vlasov equation. Then applying rigorously the waterbag reduction concept, we show that we finally recover the gyrowaterbag Equation (15), which is definitely correct even if it is not rigorously derived by using the waterbag distribution function (14) in the variables $(\mathbf{r}, v_\parallel)$. This result is not surprising since, whatever the variables we used (canonical or not) if there exists intrinsically a Liouville theorem, the latter can be expressed in such variables. Another way to understand the Liouville theorem (17) is that it is equivalent to the following conservation law satisfied by the Jacobian J:

$$\partial_t J + \nabla \cdot (J\mathbf{F}_\mathbf{r}) + \partial_{v_\parallel}(J\mathrm{F}_{v_\parallel}) = 0.$$

This conservation law can be easily recovered by taking $f = 1$ in the conservative form of the gyrokinetic-Vlasov equation (7).

Before proceeding, we need to specify the magnetic field line geometry that we consider, and describe the spatial and temporal scales of our system. Let us note that the equations written above are valid for any suitable magnetic field-line geometry. We next restrict the problem to a magnetic field having special symmetry and geometric properties described below.

C. The magnetic field line geometry and the toroidal coordinate system

In the toroidal coordinate system of Fig. 2, φ denotes the toroidal angle, θ the poloidal angle, r the minor radius of a magnetic flux surface, a the minor radius of the torus, and R_0 the major radius of the magnetic axis with $\theta = 0$ at the outside of the torus.

The expression for the magnetic field, corresponding to the assumption of concentric circular cross-section magnetic flux surface, is given in the orthonormal toroidal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ by

$$\mathbf{B} = (B_0\mathbf{e}_\varphi + B_\theta(r)\mathbf{e}_\theta)/\lambda(\theta), \tag{18}$$

where $\lambda(\theta) := 1 + \epsilon_a \cos\theta$, and with $\epsilon_a := r/R_0 \ll 1$, the inverse of the so-called aspect ratio. We then define the vector field $\mathbf{R} = R_0\lambda(\theta)(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta)$ (see Fig. 2), and $R = |\mathbf{R}| = R_0 + r\cos\theta$, where $|\cdot|$ denotes the Euclidean norm. The poloidal component of the magnetic field $B_\theta(r)$ is given by $B_\theta(r) = rB_0/(qR)$, where the so-called safety factor $q = q(r)$ is given and of order unity. Fig. 3 illustrates the geometry of a magnetic field line on a *rational magnetic flux surface* r_0 (i.e., the safety factor $q(r_0)$ takes a rational value at the particular radius r_0).

Let us define $\mathbf{b} = \mathbf{B}/B$, the unit vector tangent to the magnetic field line with $B = |\mathbf{B}|$. We then get $B_\theta/B_\varphi = b_\theta/b_\varphi = r/(qR)$, with $b_\varphi = (1 + r^2/(qR)^2)^{-1/2} = 1 + O(\epsilon_a^2) \simeq 1$ and $B = B_0/\lambda(\theta)\sqrt{1 + r^2/(qR)^2} = B_0/\lambda(\theta) + O(\epsilon_a^2) \simeq B_0/\lambda(\theta) = B_0R_0/R$.

Finally, let us recall useful expressions of the gradient, divergence, and rotational differential operators in the orthonormal toroidal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$,

$$\nabla = \mathbf{e}_r\partial_r + \mathbf{e}_\theta r^{-1}\partial_\theta + \mathbf{e}_\varphi R^{-1}\partial_\varphi, \quad \nabla \cdot \mathbf{A} = (rR)^{-1}\partial_r(rRA^r) + (rR)^{-1}\partial_\theta(RA^\theta) + R^{-1}\partial_\varphi(A^\varphi),$$

$$\nabla \times \mathbf{A} = \mathbf{e}_r(Rr)^{-1}\left\{\partial_\theta(RA^\varphi) - \partial_\varphi(rA^\theta)\right\} + \mathbf{e}_\theta R^{-1}\left\{\partial_\varphi(A^r) - \partial_r(Ra^\varphi)\right\} + \mathbf{e}_\varphi r^{-1}\left\{\partial_r(rA^\theta) - \partial_\theta A^r\right\},$$

where $(A^r, A^\theta, A^\varphi)$ are the components of the vector $\mathbf{A}$ in the orthonormal toroidal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$. Moreover, it is useful to get the expressions of the parallel and perpendicular components of the

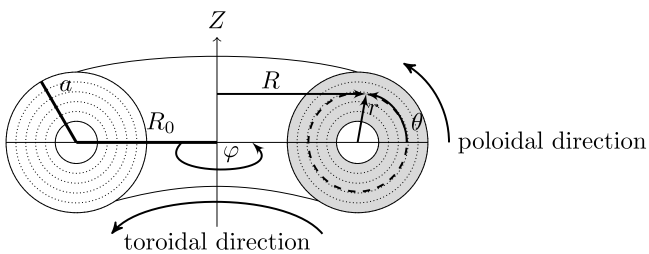

FIG. 2. Toroidal geometry.

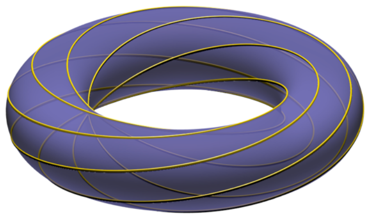

FIG. 3. Geometry of a magnetic field line (yellow) on a rational magnetic flux surface r_0 (purple).

gradient operator in the toroidal coordinate system. Since $b_\theta/b_\varphi = r/(qR)$, we get

$$\partial_\parallel = \mathbf{b}\cdot\nabla = \frac{b_\varphi}{R}\left(\partial_\varphi + \frac{1}{q}\partial_\theta\right),$$

$$\nabla_\perp = (I - \mathbf{b}\otimes\mathbf{b})\nabla = -\mathbf{b}\times(\mathbf{b}\times\nabla) = \mathbf{e}_r\partial_r + \mathbf{e}_\theta\left(-\frac{b_\theta b_\varphi}{R}\partial_\varphi + \frac{b_\varphi^2}{r}\partial_\theta\right) + \mathbf{e}_\varphi\left(\frac{b_\theta^2}{r}\partial_\varphi - \frac{b_\theta b_\varphi}{r}\partial_\theta\right),$$

where $\otimes$ denotes the tensor product.

Let us note that the geometry of our assumed magnetic field implies, after a simple calculation, the estimate

$$|\mathbf{b}.\nabla\times\mathbf{b}| \simeq 1/R, \tag{19}$$

which will be useful later on.

Remark 3. The most important magnetic confinement devices currently being developed and under construction are of axisymmetric toroidal shape, i.e., invariant under φ-angle rotation around the Z-axis, see Fig. 2. In this paper we restrict ourselves to axisymmetric toroidal geometry with concentric circular cross-section nested magnetic flux surfaces. Nevertheless, the analysis can be extended to more general shapes of axisymmetric nested toroidal magnetic flux surfaces. In that case, instead of classical toroidal coordinates we need to consider orthogonal magnetic flux coordinates and to define a local safety factor q with angle-like dependence.[60] Even if the algebra is a little more cumbersome, the method developed hereafter remains valid and can be straightforwardly applied.

D. Definition of scales and their ordering

In this section we define the different scales involved in our problem and dimensionless parameters, which fix the ratio between the different scales. The longitudinal scale $L_\parallel$ and the transverse scale $L_\perp$ are defined by

$$L_\parallel = O(qR_0), \quad L_\perp = O\left(\frac{a}{n}\right), \quad \text{so that} \quad k_\parallel = O\left(\frac{1}{qR_0}\right), \quad k_\perp = O\left(\frac{n}{a}\right),$$

where n is the toroidal mode number, and where a and R_0 are, respectively, the minor and the major radius of the axisymmetric torus (see Fig. 2). We next suppose that the longitudinal (respectively, transverse) velocity scale $\bar{v}_\parallel$ (respectively, $\bar{v}_\perp$) is of order of the ion thermal velocity $v_{th,i}$. We set $\bar{\omega} = k_\parallel\bar{v}_\parallel \simeq k_\parallel v_{th,i}$, the magnitude order of eigenfrequency of the waves, while $\rho_i = v_{th,i}/\Omega_i$ (with $\Omega_i := q_i B/m_i$) is the ion Larmor radius. We then define

$$\epsilon = \frac{1}{n}, \quad \epsilon_a = \frac{r}{R_0} = \frac{r}{R} = \frac{a}{R_0}, \quad \epsilon_\omega = \frac{\bar{\omega}}{\Omega_i}, \quad \epsilon_k = \frac{k_\parallel}{k_\perp}, \quad \epsilon_{\nabla_\perp} = \frac{1}{k_\perp a}, \quad \epsilon_\perp = k_\perp\rho_i, \quad \rho^\star = \frac{\rho_i}{a}.$$

Assuming the ordering $\epsilon < \epsilon_\perp \lesssim 1$, we then get

$$\epsilon_{\nabla_\perp} = O(\epsilon), \quad \epsilon_k = O(q^{-1}\epsilon\epsilon_a), \quad \rho^\star = O(\epsilon_\perp\epsilon_{\nabla_\perp}) \lesssim O(\epsilon), \quad k_\parallel\rho_i = O(q^{-1}\epsilon\epsilon_a\epsilon_\perp) \lesssim O(q^{-1}\epsilon\epsilon_a).$$

For microinstabilities such as the ion-temperature-gradient (ITG) instability, the physical values are typically $\epsilon_a \gtrsim 10^{-1}$ (e.g., $\epsilon_a \simeq 1/4$), $\epsilon \simeq 10^{-2}$, and $\epsilon_\omega \simeq 10^{-3}$, which lead to $\epsilon_k \simeq 10^{-3}$ and $k_\parallel\rho_i \simeq 10^{-3}$.

III. DERIVATION OF THE EIGENVALUE PROBLEM FOR THE GYROKINETIC-WATERBAG MODEL

With the eventual goal of determining the eigenelements of the linearized gyrowaterbag model, in this section, we solve the equilibrium problem and derive a two-dimensional linear integro-differential operator in a suitable form. We shall be able to reduce it into a sequence of one-dimensional integral equations by exploiting the anisotropy between parallel and transverse directions.

A. Linearization of the gyrowaterbag model

With the aim of studying the spectral properties of the operator stemming from the linearization of the gyrowaterbag model (15) and (16), we decompose the solution into a steady equilibrium state and an unsteady small perturbation. More precisely, since we assume axisymmetric magnetic flux surfaces, the steady equilibrium state remains also invariant by any rotation of the toroidal angle φ around the symmetry axis Z of the torus. Therefore we are allowed to make the following decomposition:

$$\phi(t,\mathbf{r}) = \phi_0(r,\theta) + \phi_1(t,\mathbf{r}), \quad \text{with} \quad |\phi_1| \ll 1,$$
$$v^{\pm}_{\mu b}(t,\mathbf{r}) = a^{\pm}_{\mu b}(r,\theta) + w^{\pm}_{\mu b}(t,\mathbf{r}) \quad \text{with} \quad |w^{\pm}_{\mu b}| \ll 1.$$

Here, the unknowns $(a^{\pm}_{\mu b},\phi_0)$ define the steady equilibrium state while the unknowns $(w^{\pm}_{\mu b},\phi_1)$ specify the unsteady small perturbation. Using the previous decomposition and some approximations related to the magnetic field geometry (see Appendix D), at zeroth order with respect to the perturbation, we obtain the following system (for more details, see Appendix C):

$$a^{\pm}_{\mu b}\partial_{\|}a^{\pm}_{\mu b} + \left(\frac{1}{B}\mathbf{b}\times\nabla\mathcal{J}_\mu\phi_0 + \left(\frac{\mu}{q_i} + \frac{a^{\pm^2}_{\mu b}}{\Omega_i}\right)\mathbf{b}\times\boldsymbol{\kappa}\right)\cdot\nabla_\perp a^{\pm}_{\mu b} + \left(\mathbf{b} + \frac{a^{\pm}_{\mu b}}{\Omega_i}\mathbf{b}\times\boldsymbol{\kappa}\right)\cdot\left(\frac{\mu}{m_i}\nabla B + \frac{q_i}{m_i}\nabla\mathcal{J}_\mu\phi_0\right) = 0, \tag{20}$$

$$-\nabla_\perp\cdot\left(\frac{n_{i0}}{B\Omega_i}\nabla_\perp\phi_0\right) + \frac{e\tau n_{i0}}{k_B T_{i0}}(\phi_0 - \lambda\langle\phi_0\rangle_{\|}) = \frac{2\pi\Omega_i}{q_i}\sum_{\mu b}\mathcal{A}_{\mu b}\mathcal{J}_\mu(a^+_{\mu b} - a^-_{\mu b}) - n_{i0}, \tag{21}$$

where $\boldsymbol{\kappa} := \mathbf{b}\cdot\nabla\mathbf{b}$ is the local radius-of-curvature vector of the magnetic field line. To first order — here, there is no need to expand beyond first order — we get the system

$$\partial_t w^{\pm}_{\mu b} + a^{\pm}_{\mu b}\partial_{\|}w^{\pm}_{\mu b} + \left(\frac{1}{B}\mathbf{b}\times\nabla\mathcal{J}_\mu\phi_0 + \left(\frac{\mu}{q_i} + \frac{a^{\pm^2}_{\mu b}}{\Omega_i}\right)\mathbf{b}\times\boldsymbol{\kappa}\right)\cdot\boldsymbol{\nabla}_\perp w^{\pm}_{\mu b} + w^{\pm}_{\mu b}\left(\partial_{\|}a^{\pm}_{\mu b} + 2\frac{a^{\pm}_{\mu b}}{\Omega_i}(\mathbf{b}\times\boldsymbol{\kappa})\cdot\nabla_\perp a^{\pm}_{\mu b}\right)$$
$$+ \frac{1}{B}\mathbf{b}\times\nabla\mathcal{J}_\mu\phi_1\cdot\boldsymbol{\nabla}_\perp a^{\pm}_{\mu b} + \frac{q_i}{m_i}\partial_{\|}\mathcal{J}_\mu\phi_1 + \frac{q_i}{m_i}\nabla_\perp\mathcal{J}_\mu\phi_1\cdot(\mathbf{b}\times\boldsymbol{\kappa})\frac{a^{\pm}_{\mu b}}{\Omega_i} + \frac{q_i}{m_i}\nabla_\perp\mathcal{J}_\mu\phi_0\cdot(\mathbf{b}\times\boldsymbol{\kappa})\frac{w^{\pm}_{\mu b}}{\Omega_i} = 0, \tag{22}$$

$$-\nabla_\perp\cdot\left(\frac{n_{i0}}{B\Omega_i}\nabla_\perp\phi_1\right) + \frac{e\tau n_{i0}}{k_B T_{i0}}(\phi_1 - \lambda\langle\phi_1\rangle_{\|}) = 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b}\mathcal{A}_{\mu b}\mathcal{J}_\mu(w^+_{\mu b} - w^-_{\mu b}). \tag{23}$$

B. The analytic representation of the steady equilibrium state

Here we aim at solving analytically the zeroth-order system (20) and (21), defining the steady equilibrium state. Before stating the proposition, which gives the analytic integration of equilibrium contours, we must introduce some notation and definitions.

Let $[r_{\min}, r_{\max}]$ be the radial domain and $a^{\circ}_{\mu b}(r) \in \mathscr{C}^1_b([r_{\min}, r_{\max}])$ be the given non-negative functions. We define the functions $\Lambda_{\mu b} : [r_{\min}, r_{\max}] \ni r \longmapsto \mathbb{R}^+$ as

$$\Lambda_{\mu b}(r) = \frac{2\mu B_0}{m_i a^{\circ 2}_{\mu b}(r)} \frac{r}{R_0},$$

with B_0 the maximum of the Euclidean norm of the magnetic field and R_0 the major radius of the torus (see Sec. II C). Since we will see below that a contour $a^{\pm}_{\mu b}(r,\theta)$ can be decomposed into disjoint open or closed contours with disjoint radial compact supports, for a fixed couple (μ, b) we can define the following partition of the radial domain:

$$[r_{\min}, r_{\max}] = \Delta_{O\mu b} \cup \Delta_{C\mu b}, \quad \Delta_{O\mu b} \cap \Delta_{C\mu b} = \emptyset.$$

Here

$$\Delta_{O\mu b} := \{ r \in [r_{\min}, r_{\max}] \quad \text{s.t.} \quad \Lambda_{\mu b}(r) < 1/2, \},$$
$$\Delta_{C\mu b} := \{ r \in [r_{\min}, r_{\max}] \quad \text{s.t.} \quad \Lambda_{\mu b}(r) > 1/2, \}.$$

Therefore, we define two sets

$$O = \{ (\mu, b), \quad \text{s.t.} \quad \Lambda_{\mu b}(r) < 1/2 \}, \tag{24}$$

$$C = \{ (\mu, b), \quad \text{s.t.} \quad \Lambda_{\mu b}(r) > 1/2 \}, \tag{25}$$

which correspond respectively to open and closed contours. We denote by $\mathscr{C} = O \cup C$ the set of all contours. Let us fix a couple (μ, b). We observe that the open contour (μ, b), i.e., belonging to the set O defined by (24), has radial support $\Delta_{O\mu b}$, while the closed contour (μ, b), i.e., belonging to the set C defined by (25), has radial support $\Delta_{C\mu b}$.

Using the definition of scales and of the ordering of Sec. II D, and dimensionalizing (20), we observe that the third term of (20) is of order $\rho^\star = O(\epsilon)$, while the others are of order one. Therefore, we can neglect the third term of (20), because it will be consistent with the forthcoming asymptotic analysis of order ϵ^γ, with $0 < \gamma < 1$. Neglecting that term, (20) becomes

$$a^{\pm}_{\mu b} \partial_{\parallel} a^{\pm}_{\mu b} + \left(\frac{1}{B} \mathbf{b} \times \nabla \mathcal{J}_\mu \phi_0 \right) \cdot \nabla_\perp a^{\pm}_{\mu b} + \left(\mathbf{b} + \frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \times \boldsymbol{\kappa} \right) \cdot \left(\frac{\mu}{m_i} \nabla B + \frac{q_i}{m_i} \nabla \mathcal{J}_\mu \phi_0 \right) = 0. \tag{26}$$

Within this framework, we have the following.

Proposition 1. We assume that the functions $a^{\pm}_{\mu b}(r,0)$ are symmetric so that $a^{\pm}_{\mu b}(r,0) = \pm a_{\mu b}(r,0) = \pm a^{\circ}_{\mu b}(r)$, with $a^{\circ}_{\mu b} \in \mathscr{C}^1_b([r_{\min}, r_{\max}])$ non-negative. Then (21) and (26), which govern the shape of equilibrium contours $a^{\pm}_{\mu b}$, admit the following unique solution:

$$a^{\pm}_{\mu b}(r,\theta) = \pm a^{\circ}_{\mu b}(r) \sqrt{1 + \Lambda_{\mu b}(r)(\cos\theta - 1)},$$

with $\theta \in]-\pi, \pi[$ if $(\mu, b) \in O$ or $\theta \in]-\theta_{L\mu b}, \theta_{L\mu b}[$ if $(\mu, b) \in C$. Here the limit angle $\theta_{L\mu b}$ is defined by

$$\theta_{L\mu b}(r) = \left| \arccos\left(1 - \frac{1}{\Lambda_{\mu b}(r)} \right) \right|,$$

and corresponds to the angle where the positive branch $a^{+}_{\mu b}$ and negative branch $a^{-}_{\mu b}$ of closed contours are meeting.

Proof. At the boundary of the poloidal domain (here a poloidal ring), we take the usual homogeneous Dirichlet conditions for ϕ_0. Moreover, we assume that

$$n_{i0} = \frac{2\pi\Omega_i}{q_i} \sum_{\mu b} \mathcal{A}_{\mu b} \mathcal{J}_\mu (a^{+}_{\mu b} - a^{-}_{\mu b}),$$

and that the given density n_{i0} and temperature T_{i0} have the desired regularity. We then get $\phi_0 = 0$, because the elliptic operator on the right hand side of (21) is invertible. Using $\phi_0 = 0$ into (26), and the approximation (D3) of the magnetic field line curvature (see Appendix D), Equation (20) becomes

$$\partial_\theta H^{\pm}_{\mu b} = 0, \quad \text{with} \quad H^{\pm}_{\mu b} = \frac{a^{\pm 2}_{\mu b}}{2} + \frac{\mu B}{m_i}. \tag{27}$$

Equation (27) can be easily integrated to obtain

$$a^{\pm}_{\mu b}(r,\theta) = a^{\pm}_{\mu b}(r,\theta_{0\mu b})\left(1 + \frac{2\mu(B(r,\theta_{0\mu b}) - B(r,\theta))}{m_i a^{\pm 2}_{\mu b}(r,\theta_{0\mu b})}\right)^{1/2}. \tag{28}$$

From the assumptions of Sec. II C, we can easily see that $B = B_0(1 - r/R_0 \cos\theta) + O(\epsilon_a^2)$, and thus, with the choice $\theta_{0\mu b} = 0$ for all (μ, b) (other reference points could be chosen), Equation (28) becomes

$$a^{\pm}_{\mu b}(r,\theta) = a^{\pm}_{\mu b}(r,0)\sqrt{1 + \Lambda^{\pm}_{\mu b}(r)(\cos\theta - 1)}, \quad \text{with} \quad \Lambda^{\pm}_{\mu b}(r) = \frac{2\mu B_0}{m_i a^{\pm 2}_{\mu b}(r,0)}\frac{r}{R_0}. \tag{29}$$

If the argument of the square root of (29) is positive, i.e., for $\Lambda^{\pm}_{\mu b} < 1/2$, then the corresponding contour is open in the sense that it is single-valued. But now, if $\Lambda^{\pm}_{\mu b} \geq 1/2$, then there exists a limit angle $\theta^{\pm}_{L\mu b}(r)$, given by

$$\theta^{\pm}_{L\mu b}(r) = \pm \arccos\left(1 - \frac{1}{\Lambda^{\pm}_{\mu b}(r)}\right), \tag{30}$$

such that the argument of the square root of (29) and the contour itself vanishes. Assuming now that $a^{\pm}_{\mu b}(r,0)$ are symmetric, so that $a^{\pm}_{\mu b}(r,0) = \pm a_{\mu b}(r,0) = \pm a^{\circ}_{\mu b}(r)$, with $a^{\circ}_{\mu b}(r) \geq 0$, then the contours $a^{\pm}_{\mu b}(r,\theta)$ for which $\Lambda_{\mu b} \geq 1/2$ can be connected to each other and thus form a multi-valued (double-valued) closed contour. □

Remark 4. In a field-aligned coordinates description (see Sec. III C 1), the contours must be extended in the variable θ over the whole real line by periodicity. Extension of open contours is done by periodicity of period 2π. For closed contours and each couple $(\mu, b) \in C$, we extend the contour $a_{\mu b}$ by continuity to zero in the variable θ on the set $]-\pi,\pi]\backslash]\theta^{-}_{L\mu b}, \theta^{+}_{L\mu b}[$ and next extend it to the whole real θ-line by periodicity of a period 2π.

Remark 5. Let us note that the case where $\Lambda_{\mu b} = 1/2$ is a transition point between two contour topologies (closed and open) and can be considered both closed and open. This transition point seems to lead to a loss of integrability in the equation of the perturbation, since it is an algebraic singularity of order minus one (see Sec. V B 1). Hence we suppress this pathological case in the definition of the equilibrium contours. This is not surprising, since the nature of this transition point is the same as the X-point in an autonomous one-dimensional Hamiltonian system, where the role of the separatrix is played here by the contour for which $\Lambda_{\mu b} = 1/2$.

Remark 6. Let us note that $H^{\pm}_{\mu b}$ can be interpreted as the unperturbed equilibrium Hamiltonian, associated to the steady equilibrium contours $a^{\pm}_{\mu b}$. Therefore, (27) can be interpreted as a conservation law, which expresses the conservation of the unperturbed equilibrium Hamiltonian on magnetic flux surfaces (here, axisymmetric nested tori with circular cross-section). Equation (20) can be seen as a first-order transport equation of the form

$$F_\theta\left(r,\theta,a^{\pm}_{\mu b}\right)\partial_\theta a^{\pm}_{\mu b} + F_r\left(r,\theta,a^{\pm}_{\mu b}\right)\partial_r a^{\pm}_{\mu b} = S\left(r,\theta,a^{\pm}_{\mu b}\right), \tag{31}$$

with the given initial conditions $r(t_0) = r_0$, $\theta(t_0) = \theta_0$, and $a^{\pm}_{\mu b}(r(t_0),\theta(t_0)) = a^{\pm}_{\mu b}(r_0,\theta_0)$. Equation (31) can be solved by the characteristic curves method (i.e., $\{d_t r = F_r;\ d_t\theta = F_\theta;\ d_t a^{\pm}_{\mu b} = S\}$), as long as the characteristics are regular enough and do not cross.

Remark 7. By neglecting terms of order $\rho^\star$ in the first-order Equation (20) and choosing symmetric radial profiles for $a^{\pm}_{\mu b}(r,0)$ (i.e., $a^{\pm}_{\mu b}(r,0) = \pm a^{\circ}_{\mu b}(r)$), we obtain symmetric closed contours, i.e., a symmetry between the positive and negative branches of a closed contour. Therefore the projection on the poloidal plane of the level lines (in absolute value) of the positive and negative branch of a closed contour, coincide. On the contrary, when we solve the initial value problem (20) or (31), the level lines (in absolute value) of the negative and positive branch of a closed contour will no more be symmetric. Consequently, projections on the poloidal plane will not coincide and thus poloidal projections of level lines will form some kind of banana-shaped orbits with non-zero width. The symmetric case corresponds to banana orbits with zero banana width.

C. System for the perturbation

We will define a *well-suited* system for the perturbation (22) and (23) that will facilitate the asymptotic analysis in Sec. III C 3, by using a field-aligned coordinate system and the ballooning-eikonal representation presented respectively in Secs. III C 1 and III C 2.

1. Field-aligned coordinate system

Assuming that the given radial function $q : [r_{\min}, r_{\max}] \longrightarrow \mathbb{R}^+$ is such that $q' > 0$ (see Sec. II C), the coordinate system aligned with the magnetic field lines reads

$$\begin{aligned} x &= r - r_0, & & \qquad & \partial_r &= \partial_x - q'\eta\partial_\alpha,\\ \eta &= \theta, & & \text{which implies} & \partial_\theta &= \partial_\eta - q\partial_\alpha,\\ \alpha &= \varphi - q(r)\theta, & & & \partial_\varphi &= \partial_\alpha. \end{aligned}$$

Here the constant radius r_0 is a reference rational magnetic flux surface, and (r,θ,φ) is the toroidal coordinate system (see Sec. II C for more details). Using the previous field-aligned coordinate system, the parallel and perpendicular gradient operators read in the orthonormal toroidal basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$

$$\begin{aligned} \partial_\parallel &= \frac{b_\varphi}{qR}\partial_\eta,\\ \nabla_\perp &= \mathbf{e}_r\left(-q'\eta\partial_\alpha + \partial_x\right) + \mathbf{e}_\theta\left(\mathfrak{g}_{11}\partial_\alpha + \mathfrak{g}_{12}\partial_\eta\right) + \mathbf{e}_\varphi\left(\mathfrak{g}_{21}\partial_\alpha + \mathfrak{g}_{22}\partial_\eta\right), \end{aligned}$$

with the matrix coefficients $\mathfrak{g}_{11} = -\mathfrak{g}_{12}\mathfrak{q}$, $\mathfrak{g}_{21} = -\mathfrak{g}_{22}\mathfrak{q}$, $\mathfrak{g}_{12} = b_\varphi^2/r$, $\mathfrak{g}_{22} = -b_\theta b_\varphi/r$, and the safety-factor-like $\mathfrak{q} = (r^2 + q^2R^2)/(qR^2)$. Since $q' > 0$ ($q > 0$), we can use as radial variable, either r, q or x whose domains of definition are respectively given by $[r_{\min}, r_{\max}]$, $[x_{\min}, x_{\max}]$, and $[q_{\min}, q_{\max}]$, with $x_{\min} = r_{\min} - r_0$, $x_{\max} = r_{\max} - r_0$, $q_{\min} = q(r_{\min})$, and $q_{\max} = q(r_{\max})$.

2. Ballooning transformation and eikonal representation

Description of the ballooning transformation. In this section we briefly present the ballooning transformation, which is commonly used in toroidally confined plasmas to represent a field perturbation in an axisymmetric system.[59,61,60,23,78] The first step of this method is to transform the θ-periodic space into an unbounded "covering space" in the variable η, which has the sense of a coordinate along the magnetic field lines. The second step takes advantage of the anisotropy between the η-parallel and the α-transverse directions to use eikonal analysis. Since we have assumed axisymmetric equilibrium magnetic flux surfaces, toroidal Fourier modes in the φ-variable with wavenumber $n \in \mathbb{Z}$ are still eigenmodes in the toroidal φ-direction, while this is no more the case for the poloidal Fourier modes in the θ-variable with wavenumber $m \in \mathbb{Z}$. In other words the eigenmode envelope in the (r,θ)-variables satisfies a truly two-dimensional quasilinear integro-differential equation in the (r,θ)-variables, with nonlinear (resp. convolution) terms in the variable θ (resp. m). Let us consider an arbitrary perturbation $\phi(t,\mathbf{r})$. The periodicity of the perturbation in the toroidal (θ,φ)-variables allows us to use a Fourier decomposition

$$\phi = \phi(t,\mathbf{r}) = \sum_{n\in\mathbb{Z}} \Phi_n(t,r,\theta)\exp(\mathrm{i}n\varphi) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \Phi_{mn}(t,r)\exp(\mathrm{i}(n\varphi - m\theta)),$$

where

$$\Phi_{mn}(t,r) = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi\, \phi(t,\mathbf{r})\exp(\mathrm{i}(m\theta - n\varphi)).$$

In the field-aligned variables $\mathbf{r} = (x,\eta,\alpha)$, since $\phi = \phi(t,r,\theta,\varphi) = \tilde{\phi}(t,x,\eta,\alpha) = \tilde{\phi}(t,x,\eta,\varphi - q\theta)$, the 2π-periodicity of ϕ in φ implies the 2π-periodicity of $\tilde{\phi}$ in α. Therefore in the field-aligned variables we can use a Fourier decomposition along the α-variable, i.e.,

$$\phi = \tilde{\phi}(t,\mathbf{r}) = \sum_{n\in\mathbb{Z}} \widetilde{\Phi}_n(t,x,\eta)\exp(\mathrm{i}n\alpha),$$

where the nth Fourier mode Φ_n in φ-variable is linked to the nth Fourier mode $\widetilde{\Phi}_n$ in the α-variable by $\widetilde{\Phi}_n(t,x,\theta) = \Phi_n(t,r,\theta)\exp(\mathrm{i}nq\theta)$. But the n-th Fourier mode $\widetilde{\Phi}_n(t,x,\eta)$ is generally not periodic in η since its periodicity depends on rationality of the safety factor q.

The inverse Laplace transform and the residue theorem allow us to obtain the following spectral decomposition:

$$\phi(t,\mathbf{r}) = \sum_{n\in\mathbb{Z}}\sum_{\omega\in\mathscr{S}_n} \exp(-\mathrm{i}\omega t)\Phi_{\omega n}(r,\theta)\exp(\mathrm{i}n\varphi), \tag{32}$$

where $\mathscr{S}_n$ is the spectrum that we still have to determine. Let us note that we have $\widetilde{\Phi}_{\omega n}(x,\theta) = \Phi_{\omega n}(r,\theta)\exp(\mathrm{i}nq\theta)$. We now assume that for every $\omega \in \mathbb{C}$ and $n \in \mathbb{Z}$, there exists a function $\widehat{\Phi}_{\omega n} \in L^2([r_{\min}, r_{\max}], \mathbb{R}_\eta)$, such that $\Phi_{\omega n} = \widehat{\Phi}_{\omega n} * \Delta_{2\pi}$, where $\Delta_{2\pi}$ is the 2π-periodic Dirac comb in the θ-variable. Therefore, using Poisson's sum formula, we get

$$\Phi_{\omega n}(r,\theta) = \sum_{m\in\mathbb{Z}} \exp(-\mathrm{i}m\theta)\frac{1}{2\pi}\int_{\mathbb{R}} d\eta\, \widehat{\Phi}_{\omega n}(r,\eta)\exp(\mathrm{i}m\eta),$$

which by identification leads to

$$\Phi_{\omega mn}(r) = \frac{1}{2\pi}\int_{\mathbb{R}} d\eta\, \widehat{\Phi}_{\omega n}(r,\eta)\exp(\mathrm{i}m\eta). \tag{33}$$

Since in an axisymmetric toroidal confinement system, micro-instabilities (flute-like perturbations) develop small perpendicular scales comparatively to parallel ones, the natural technique for such problem with disparate length scales is the eikonal or WKB decomposition. Therefore, with $n \gg 1$, we use the following eikonal form:

$$\widehat{\Phi}_{\omega n}(r,\eta) = \widehat{\phi}_{\omega n}(\eta; q, \theta_k(q))\exp(\mathrm{i}nS(\eta,q,\theta_k(q))). \tag{34}$$

Here, the eikonal S is given by

$$S(\eta,q,\theta_k(q)) = -q\eta + \int dq\, \theta_k(q), \tag{35}$$

where θ_k denotes a normalized radial wavenumber conjugate to the radial variable q.

Substituting the eikonal (35) into the Fourier mode (33), we observe two dual relationships. First, the variable η, which determines the global mode azimuthal structure (slow poloidal variation), is dual to the variable $nq - m \simeq x/d_{mn}$, which provides the local radial structure (rapid radial variation). (Previously we have used the definitions $x = r - r_{mn}$ and $d_{mn} = 1/(nq'(r_{mn}))$, where d_{mn} corresponds to the distance between neighboring rational magnetic flux surfaces and where the *rational magnetic flux surface* r_{mn} associated to the mode (m,n) is defined by $nq(r_{mn}) = m$.) Second, the variable θ_k, dual to nq, provides the global radial structure (slow radial variation). The global azimuthal structure is given by the envelope $\widehat{\phi}_{\omega n}(\eta; q, \theta_k(q))$, which depends parametrically on (ω, n, q, θ_k), and is such that $\partial_\eta \widehat{\phi}_{\omega n} \simeq \partial_\eta S \simeq O(1)$. We shall return to the definition of the different space scales involved in our problem in Sec. IV.

Because of the safety factor q, in a sheared system (i.e., with q non-constant with respect to the radius r) flute-like perturbations (i.e., $k_\parallel \ll k_\perp$) can occur only near rational magnetic flux surfaces, which are isolated and defined by $m = nq$.[60] It is easy to prove that on toroidal magnetic flux surfaces, pure flute perturbations (i.e., $k_\parallel = 0$) can only occur on rational magnetic flux surfaces. Indeed, to have different values of the perturbation in the transverse direction to the magnetic field on a given magnetic flux surface there must exist distinguishable field lines and irrational magnetic flux surfaces are densely covered by a single field line. Therefore, for flute-like disturbances such as the ITG instability, the poloidal number m and toroidal number n are not really independent. As a consequence, the spectrum $\mathscr{S}_n$ associated to such eigenmodes depends only on the toroidal number n. Therefore, using (34), and Poisson's sum formula we get

$$\begin{aligned}\phi(t,\mathbf{r}) &= \sum_{(n,m)\in\mathbb{Z}^2}\sum_{\omega\in\mathscr{S}_n} \exp(-\mathrm{i}\omega t)\exp(-\mathrm{i}m\theta)\frac{1}{2\pi}\int_{\mathbb{R}} d\eta\,\exp(\mathrm{i}m\eta)\widehat{\phi}_{\omega n}(\eta;q,\theta_k)\exp\left(\mathrm{i}n\left(\alpha+\int dq\,\theta_k\right)\right)\\ &= \sum_{(n,\ell)\in\mathbb{Z}^2}\sum_{\omega\in\mathscr{S}_n} \exp(-\mathrm{i}\omega t)\widehat{\phi}_{\omega n}(\theta+2\pi\ell;q,\theta_k)\exp\left(\mathrm{i}n\left(\alpha-2\pi\ell q+\int dq\,\theta_k\right)\right).\end{aligned} \tag{36}$$

Expansion (36) can either be written in the toroidal variables $\mathbf{r} = (r,\theta,\varphi)$ (i.e., $\phi = \phi(t,r,\theta,\varphi)$) or in the field-aligned variables $\mathbf{r} = (x,\eta,\alpha)$ (i.e., $\phi = \tilde{\phi}(t,x,\eta,\alpha)$) of Sec. III C 1. Expansion (36), which can been viewed as a generalized transform analogous to the Fourier or Laplace transforms, is called the ballooning transformation. It was introduced by tokamak physicists to get a pseudo-spectral decomposition of the flute-like modes, strongly localized poloidally but not radially, and which can be seen as coupled set of modes with different helicities and nearly equal amplitude.

Remark 8. There exist different versions — with different names — of the ballooning transformation, whose mathematical definition is not always fully specified. The mathematical treatment of short-wavelength toroidal eigenmodes using ballooning transformation is similar to the Bloch analysis of lattices in solid state physics.[73,72,70] *An attempt for laying the mathematical foundations of the ballooning transformation, where questions of existence, uniqueness, and inversion of such transform are discussed and somehow cleared up, can be found in Refs. 59, 61, 60, 23, and 78.*

Remark 9. Let us note that the inhomogeneous normalized radial wavenumber θ_k can also be interpreted as a differential operator, which we denote by Θ_k. Indeed, let us define the complex amplitude $A(q)$ by

$$A(q) = \exp\left(\mathrm{i}n\int_{q_0}^{q} d\tilde{q}\,\theta_k(\tilde{q})\right)A(q_0) \quad \textit{or} \quad A(x) = \exp\left(\mathrm{i}n\int_{x_0}^{x} d\tilde{x}\,q'(\tilde{x})\theta_k(\tilde{x})\right)A(x_0).$$

The differential operators Θ_k and $\mathcal{T}_q$ are defined respectively by

$$\Theta_k = -\frac{\mathrm{i}}{n}\partial_q = -\frac{\mathrm{i}}{nq'}\partial_x \quad \textit{and} \quad \mathcal{T}_q = \exp\left(\mathrm{i}n\int_{q_0}^{q} d\tilde{q}\,\Theta_k\right).$$

Assuming now that $A \in \mathscr{C}^\infty([q_{\min},q_{\max}])$, we get

$$\mathcal{T}_q A(q_0) = \exp\left(\mathrm{i}n\int_{q_0}^{q} d\tilde{q}\,\Theta_k\right)A(q_0) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}(q-q_0)^\ell\partial_q^\ell A(q_0) = A(q).$$

Therefore, the operator $\mathcal{T}_q$ can be interpreted as a translation operator, which gives the state $A(q)$ at q, when acting on the reference state $A(q_0)$ at $q = q_0$. Thus, we can indifferently write

$$A(q) = \exp\left(\mathrm{i}n\int_{q_0}^{q} d\tilde{q}\,\theta_k(\tilde{q})\right)A(q_0) = \exp\left(\mathrm{i}n\int_{q_0}^{q} d\tilde{q}\,\Theta_k\right)A(q_0).$$

The action of the operator Θ_k on the complex amplitude A is thus given by

$$\begin{aligned}\Theta_k A &= \theta_k A = A\theta_k,\\ (\Theta_k)^2 A &= A\Theta_k\theta_k + \theta_k\Theta_k A = A\left(\theta_k^2 - \frac{i}{n}\partial_q\theta_k\right),\\ (\Theta_k)^\ell A &= A\left(\theta_k^\ell + \frac{\ell(\ell-1)}{2\mathrm{i}n}\theta_k^{\ell-2}\partial_q\theta_k + O(n^{-2})\right).\end{aligned}$$

The inhomogeneous normalized radial wavenumber should be considered as a function $\theta_k(q)$ (respectively, a differential operator Θ_k) when it is placed at the right (respectively, left) of the complex amplitude A.

Use of the ballooning transformation. Here we describe the action of a linear integro-differential operator and the gyroaverage operator in the ballooning representation. Let us start with the gyroaverage operator. From the definition (4) of the gyroaverage operator $\mathcal{J}_\mu$, we obtain

$$(\mathcal{J}_\mu \phi)(t,r,\theta,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} d\zeta\ \phi(t,\bar{r}(\zeta),\bar{\theta}(\zeta),\bar{\varphi}(\zeta)),$$

with[100]

$$\bar{r}(\zeta) = r - \frac{v_\perp}{\Omega_i}\sin\zeta, \quad \bar{\theta}(\zeta) = \theta - \frac{v_\perp}{\Omega_i}\frac{1}{r}\cos\zeta, \quad \bar{\varphi}(\zeta) = \varphi - \frac{v_\perp}{\Omega_i}\frac{r}{qR^2}\cos\zeta.$$

Therefore, using the ballooning expansion (36), we get

$$\begin{aligned}(\mathcal{J}_\mu \phi)(t,r,\theta,\varphi) &= \sum_{(n,\ell)\in\mathbb{Z}^2}\sum_{\omega\in\mathcal{S}_n} \exp(-\mathrm{i}\omega t)\\ &\quad \frac{1}{2\pi}\int_0^{2\pi} d\zeta\ \exp\left(\mathrm{i}n\left\{\bar{\varphi} - q(\bar{r})(\bar{\theta}+2\pi\ell) + \int dq(\bar{r})\,\theta_k(\bar{x})\right\}\right)\widehat{\phi}_{\omega n}(\bar{\theta}+2\pi\ell;\bar{x},\theta_k(\bar{x}))\\ &= \sum_{(n,\ell)\in\mathbb{Z}^2}\sum_{\omega\in\mathcal{S}_n}\exp(-\mathrm{i}\omega t)\exp\left(\mathrm{i}n\left\{\varphi - q(r)(\theta+2\pi\ell) + \int dq(r)\,\theta_k(x)\right\}\right)\mathfrak{J}_\mu(\theta+2\pi\ell)\widehat{\phi}_{\omega n}(\theta+2\pi\ell;x,\theta_k(x)),\end{aligned} \tag{37}$$

with $\bar{x}(\zeta) = x - \frac{v_\perp}{\Omega_i}\sin\zeta$, and where

$$\begin{aligned}\mathfrak{J}_\mu(\theta+2\pi\ell)\widehat{\phi}_{\omega n}(\eta;x,\theta_{1k}) &= \frac{1}{2\pi}\int_0^{2\pi} d\zeta\ \exp\left(\mathrm{i}\frac{n}{r}q\left(r - \frac{v_\perp}{\Omega_i}\sin\zeta\right)\frac{v_\perp}{\Omega_i}\cos\zeta\right)\\ &\quad \exp\left(\mathrm{i}n\int dx\ \{q'(r-(v_\perp/\Omega_i)\sin\zeta)\theta_k(x-(v_\perp/\Omega_i)\sin\zeta) - q'(r)\theta_k(x)\}\right)\\ &\quad \exp\left(\mathrm{i}n\left[-\frac{v_\perp r}{\Omega_i qR^2}\cos\zeta - (\theta+2\pi\ell)\{q(r-(v_\perp/\Omega_i)\sin\zeta) - q(r)\}\right]\right)\\ &\quad \widehat{\phi}_{\omega n}\left(\eta - \frac{v_\perp}{\Omega_i r}\cos\zeta;\ x - \frac{v_\perp}{\Omega_i}\sin\zeta, \theta_k\left(x - \frac{v_\perp}{\Omega_i}\sin\zeta\right)\right).\end{aligned}$$

Let us now describe the action of a certain types of linear integro-differential operators $\mathcal{L}$, on a perturbed quantity ϕ. Here, we assume that the linear integro-differential operator takes the form

$$\mathcal{L} = \sum_{\mu b}\mathcal{J}_\mu\mathcal{L}_{\mu b}(x,\theta,\partial_t,\partial_\varphi^j,\partial_r^p,\partial_\theta^m)\mathcal{J}_\mu,$$

where $\mathcal{L}_{\mu b}$ stands for a linear differential operator in which the dependence on the variables (x,θ) is governed by the unperturbed (equilibrium) solution, which is periodic in θ-variable. Using (37), we then obtain

$$\begin{aligned}\mathcal{L}\phi &= \sum_{\mu b}\mathcal{J}_\mu\mathcal{L}_{\mu b}(x,\theta,\partial_t,\partial_\varphi^j,\partial_r^p,\partial_\theta^m)\mathcal{J}_\mu\sum_{(n,l)\in\mathbb{Z}^2}\sum_{\omega\in\mathcal{S}_n} e^{-\mathrm{i}\omega t}e^{\mathrm{i}n(\varphi - q(\theta+2\pi\ell)+\int dq\,\theta_k)}\widehat{\phi}_{\omega n}(\theta+2\pi\ell;x,\theta_k)\\ &= \sum_{\mu b}\mathcal{J}_\mu\mathcal{L}_{\mu b}(x,\theta,\partial_t,\partial_\varphi^j,\partial_r^p,\partial_\theta^m)\\ &\quad\times\sum_{(n,l)\in\mathbb{Z}^2}\sum_{\omega\in\mathcal{S}_n} e^{-\mathrm{i}\omega t}e^{\mathrm{i}n(\varphi - q(\theta+2\pi\ell)+\int dq\,\theta_k)}\mathfrak{J}_\mu(\theta+2\pi\ell)\widehat{\phi}_{\omega n}(\theta+2\pi\ell;x,\theta_k)\\ &= \sum_{\mu b}\mathcal{J}_\mu\sum_{(n,l)\in\mathbb{Z}^2}\sum_{\omega\in\mathcal{S}_n} e^{-\mathrm{i}\omega t}e^{\mathrm{i}n(\varphi - q(\theta+2\pi\ell)+\int dq\,\theta_k)}\\ &\quad\mathcal{L}_{\mu b}\left(x,\theta,-\mathrm{i}\omega,(\mathrm{i}n)^j,(\partial_x - \mathrm{i}nq'[\theta+2\pi\ell-\theta_k])^p,(\partial_\theta - \mathrm{i}nq)^m\right)\mathfrak{J}_\mu(\theta+2\pi\ell)\widehat{\phi}_{\omega n}(\theta+2\pi\ell;x,\theta_k)\end{aligned}$$

$$= \sum_{(n,l)\in\mathbb{Z}^2} \sum_{\omega\in\mathscr{S}_n} e^{-\mathrm{i}\omega t} e^{\mathrm{i}n(\varphi - q(\theta+2\pi\ell)+\int dq\,\theta_k)} \sum_{\mu b} \mathfrak{I}_\mu(\theta+2\pi\ell)$$
$$\mathcal{L}_{\mu b}\left(x,\theta,-\mathrm{i}\omega,(\mathrm{i}n)^j,(\partial_x - \mathrm{i}nq'(\theta+2\pi\ell-\theta_k))^p,(\partial_\theta - \mathrm{i}nq)^m\right)\mathfrak{I}_\mu(\theta+2\pi\ell)\widehat{\phi}_{\omega n}(\theta+2\pi\ell;x,\theta_k)$$
$$:= 0.$$

After making the change of variables $\eta = \theta + 2\pi\ell$ ($\partial_\eta = \partial_\theta$) and using the periodicity (of the steady equilibrium state) in the θ-variable, we get

$$\sum_{(n,l)\in\mathbb{Z}^2} \sum_{\omega\in\mathscr{S}_n} e^{-\mathrm{i}\omega t+\mathrm{i}n(\alpha+\int dq\,\theta_k)} \sum_{\mu b} \mathfrak{I}_\mu\mathcal{L}_{\mu b}\left(x,\eta,-\mathrm{i}\omega,(\mathrm{i}n)^j,(\partial_x - \mathrm{i}nq'(\eta-\theta_k))^p,(\partial_\eta - \mathrm{i}nq)^m\right)\mathfrak{I}_\mu\widehat{\phi}_{\omega n}(\eta;x,\theta_k) = 0. \quad (38)$$

Since the integrand in (38) does not depend on ℓ, we get the integro-differential equation

$$\mathfrak{I}_\mu\mathcal{L}_{\mu b}\left(x,\eta,-\mathrm{i}\omega,(\mathrm{i}n)^j,(\partial_x - \mathrm{i}nq'(\eta-\theta_k))^p,(\partial_\eta - \mathrm{i}nq)^m\right)\mathfrak{I}_\mu\widehat{\phi}_{\omega n}(\eta;x,\theta_k) = 0,$$

where $\omega\in\mathscr{S}_n$ is the eigenfrequency and $\widehat{\phi}_{\omega n}$ is the eigenmode.

3. The well-suited system for the perturbation

We are now ready to reformulate the equations for the perturbations (22) and (23) as a well-suited system, by using the field-aligned coordinates of Sec. III C 1 and the ballooning-eikonal representation of Sec. III C 2. We emphasize that this section is only devoted to rewriting an equivalent system for (22) and (23) and not to performing its asymptotic analysis, even if we already take into account the anisotropy of the problem through the ballooning-eikonal representation in the field-aligned coordinate system.

Substituting the ballooning representation (36) — see Sec. III C 2 for its use — into the system (22) and (23), choosing the field-aligned coordinate system of Sec. III C 1 and using $\phi_0 = 0$, after some algebra, we obtain for every contour $(\mu,b)\in\mathscr{C}$ the two-dimensional integro-differential equations

$$\mathcal{L}^{\pm}_{\mu b\omega n}(\omega,x,\eta,\eta-\theta_k,\partial_x,\partial_\eta)w^{\pm}_{\mu b\omega n} + \mathcal{M}^{\pm}_{\mu b\omega n}(\omega,x,\eta,\eta-\theta_k,\partial_x,\partial_\eta)\mathfrak{I}_\mu\phi_{1\omega n} = 0. \quad (39)$$

In (39) the linear differential operators $\mathcal{L}^{\pm}_{\mu b\omega n}$ and $\mathcal{M}^{\pm}_{\mu b\omega n}$ are defined by

$$\begin{aligned}\mathcal{L}^{\pm}_{\mu b\omega n} = &\left\{-\mathrm{i}\omega + \mathrm{i}n\left(\frac{\mu}{q_i}+\frac{a^{\pm}_{\mu b}}{\Omega_i}\right)\frac{b_\varphi}{R}\left(q'(\eta-\theta_k)\sin\eta + \mathfrak{q}\frac{\cos\eta}{r}\right)\right.\\ &\left.+\frac{b_\varphi}{qR}\partial_\eta a^{\pm}_{\mu b} - 2\frac{a^{\pm}_{\mu b}}{\Omega_i}\frac{b_\varphi}{R}\left(\sin\eta\,\partial_x a^{\pm}_{\mu b} - \frac{\cos\eta}{r}\partial_\eta a^{\pm}_{\mu b}\right)\right\}\\ &-\left\{\left(\frac{\mu}{q_i}+\frac{a^{\pm}_{\mu b}}{\Omega_i}\right)\frac{b_\varphi}{R}\frac{\cos\eta}{r} - a^{\pm}_{\mu b}\frac{b_\varphi}{qR}\right\}\partial_\eta - \left\{\left(\frac{\mu}{q_i}+\frac{a^{\pm}_{\mu b}}{\Omega_i}\right)\frac{b_\varphi}{R}\sin\eta\right\}\partial_x,\end{aligned} \quad (40)$$

$$\begin{aligned}\mathcal{M}^{\pm}_{\mu b\omega n} = \mathrm{i}nb_\varphi&\left\{\frac{1}{rB}\left(\mathfrak{q}\partial_x a^{\pm}_{\mu b} - q'(\eta-\theta_k)\partial_\eta a^{\pm}_{\mu b}\right) + \frac{q_i}{m_i}\frac{a^{\pm}_{\mu b}}{\Omega_i}\frac{1}{R}\left(q'(\eta-\theta_k)\sin\eta + \mathfrak{q}\frac{\cos\eta}{r}\right)\right\}\\ &+\left\{\frac{q_i}{m_i}\frac{b_\varphi}{qR} - \frac{b_\varphi}{rB}\partial_x a^{\pm}_{\mu b} - \frac{q_i}{m_i}\frac{a^{\pm}_{\mu b}}{\Omega_i}\frac{b_\varphi}{R}\frac{\cos\eta}{r}\right\}\partial_\eta + \left\{\frac{b_\varphi}{rB}\partial_\eta a^{\pm}_{\mu b} - \frac{q_i}{m_i}\frac{a^{\pm}_{\mu b}}{\Omega_i}\frac{b_\varphi}{R}\sin\eta\right\}\partial_x,\end{aligned} \quad (41)$$

while the linear integral operator $\mathfrak{I}_\mu$ (see paragraph "Use of the ballooning transform" of Sec. III C 2) is defined by

$$\mathfrak{I}_\mu(\eta')\widehat{\phi}_{\omega n}(\eta;x,\theta_k) = \frac{1}{2\pi}\int_0^{2\pi} d\zeta\ \exp\left(\mathrm{i}\frac{n}{r}q\left(r-\frac{v_\perp}{\Omega_i}\sin\zeta\right)\frac{v_\perp}{\Omega_i}\cos\zeta\right)$$
$$\exp\left(in\int dx\ \{q'(r-(v_\perp/\Omega_i)\sin\zeta)\theta_k(x-(v_\perp/\Omega_i)\sin\zeta)-q'(r)\theta_k(x)\}\right)$$
$$\exp\left(in\left[-\frac{v_\perp r}{\Omega_i q R^2}\cos\zeta-\eta'\{q(r-(v_\perp/\Omega_i)\sin\zeta)-q(r)\}\right]\right)$$
$$\widehat{\phi}_{\omega n}\left(\eta-\frac{v_\perp}{\Omega_i r}\cos\zeta;\ x-\frac{v_\perp}{\Omega_i}\sin\zeta,\theta_k\left(x-\frac{v_\perp}{\Omega_i}\sin\zeta\right)\right). \quad (42)$$

As it is commonly done for the quasi-neutrality equation (5), the transverse direction to the magnetic field (i.e., $\mathbf{b}^\perp$) in the differential term of Equation (23) is approximated by the transverse direction to $\mathbf{e}_\varphi$, i.e., the poloidal cross-section of the torus, which belongs to $\mathbf{e}_\varphi^\perp$. Using this approximation, the field-aligned coordinates and the ballooning-eikonal representation (see Sec. III C 1 and Sec. III C 2) Equation (23) becomes

$$\mathcal{Q}_{\omega n}(\omega,x,\eta,\eta-\theta_k,\theta_k',\partial_x,\partial_\eta)\phi_{1\omega n} = 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}(\mathfrak{I}_\mu w^+_{\mu b\omega n}-\mathfrak{I}_\mu w^-_{\mu b\omega n}), \quad (43)$$

where

$$\mathcal{Q}_{\omega n} = \left\{-\frac{1}{rR}\partial_x rR\frac{n_{i0}}{\Omega_i B}\partial_x + \mathrm{i}nq'(\eta-\theta_k)\frac{n_{i0}}{\Omega_i B}\partial_x - \frac{1}{r^2R}\partial_\eta R\frac{n_{i0}}{\Omega_i B}\partial_\eta + \frac{e\tau n_{i0}}{k_B T_{i0}} + \frac{n_{i0}}{\Omega_i B}\left(\frac{n^2q^2}{r^2}+n^2q'^2(\eta-\theta_k)^2\right)\right.$$
$$\left. in\left(\frac{q}{r^2}\frac{n_{i0}}{\Omega_i B}+q'(\eta-\theta_k)\frac{1}{rR}\partial_x\left(rR\frac{n_{i0}}{\Omega_i B}\right)+\frac{q}{r^2R}\partial_\eta\left(R\frac{n_{i0}}{\Omega_i B}\right)+\frac{n_{i0}}{\Omega_i B}(q''(\eta-\theta_k)-q'\theta_k')\right)\right\}. \quad (44)$$

Introducing for all $(\mu,b)\in\mathscr{C}$, the perturbed Hamiltonian $h^\pm_{\mu b\omega n}$, which is defined as

$$h^\pm_{\mu b\omega n} = a^\pm_{\mu b}w^\pm_{\mu b\omega n} + \frac{q_i}{m_i}\mathfrak{I}_\mu\phi_{1\omega n},$$

the system for the perturbation (39)-(43) is recast as

$$\mathcal{L}^\pm_{\mu b\omega n}\left(\frac{h^\pm_{\mu b\omega n}}{a^\pm_{\mu b}}\right) + \mathcal{M}^\pm_{\mu b\omega n}\mathfrak{I}_\mu\phi_{1\omega n} - \frac{q_i}{m_i}\mathcal{L}^\pm_{\mu b\omega n}\left(\frac{\mathfrak{I}_\mu\phi_{1\omega n}}{a^\pm_{\mu b}}\right) = 0, \quad (45)$$

$$\left\{\mathcal{Q}_{\omega n} + 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\mathfrak{I}_\mu\left(\frac{1}{a^+_{\mu b}}-\frac{1}{a^-_{\mu b}}\right)\mathfrak{I}_\mu\right\}\phi_{1\omega n} = 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\mathfrak{I}_\mu\left(\frac{h^+_{\mu b\omega n}}{a^+_{\mu b}}-\frac{h^-_{\mu b\omega n}}{a^-_{\mu b}}\right). \quad (46)$$

IV. ASYMPTOTIC ANALYSIS

In this section we perform an asymptotic analysis of the system (45) and (46), by taking advantage of the anisotropy between the parallel and the transverse directions. This, in turn, leads to a reduction of the two-dimensional integro-differential equations (45) and (46) into a sequence of one-dimensional integral equations coupled to a one-dimensional non-self-adjoint Schrödinger equation. In Sec. IV A, using the definition of scales and of the ordering of Sec. II D, we perform the asymptotic expansions in the small parameter ϵ^γ ($\gamma\in]0,1[$). This requires expansions in power of ϵ^γ of not only the electric potential but also the linearized integro-differential operator and the eigenvalues. We next recast the zeroth-order problem, arising from the asymptotic analysis, into a sequence of one-dimensional integral equations using Proposition 2 of Sec. IV B. In Sec. IV C, we solve the first-order problem through Propositions 3–5. This leads to the determination of the ballooning angle. In Sec. IV D, we then rewrite the second-order problem as a one-dimensional

Schrödinger equation using Proposition 6. In Sec. IV F, we give an algorithm, based on the asymptotic analysis, to solve the eigenvalue problem for the two-dimensional integro-differential gyrowaterbag operator. In order to simplify calculations, at first, we consider the particular case where there is no gyroaverage operator, i.e., we set $\mathfrak{J}_\mu = 1$ in (45) and (46). In Sec. IV E we establish Proposition 7 which allows us to handle the general case of (45) and (46).

A. Asymptotic expansion

We now assume that we have two scales of variation in the poloidal direction, one fast and the other slow; and three scales of variation in the radial direction, one fast, a second slow, and a third intermediate length scale. The fast poloidal scale of length order $1/(nq)$ is represented by the term "$nq\eta$" in the eikonal of the ballooning representation (see Sec. III C 2), while the slow poloidal scale of length order 1 is associated to the η-variation of the envelope $\widehat{\phi}_{1\omega n}$. Since we assume $\theta_k = O(1)$, the fast radial scale, of length order $d = 1/(nq')$ and which can be interpreted as the typical distance between two rational magnetic flux surfaces, is taken into account in the eikonal term (35) of the ballooning representation (see Sec. III C 2). The slow radial scale, of length order a and which is the scale of variation of the density n_{i0} and the temperature T_{i0}, is associated to the x-variation of the envelope $\widehat{\phi}_{1\omega n}$. The eigenmode envelope radial scales which are amenable to our asymptotic expansion are scales intermediate between the two aforementioned ones. For this we assume a radial variation on a length scale of order $n^{-\sigma}a$ with $0 < \sigma < 1$. In order to take into account this scale of variation, which gives the radial extension of the searched eigenmodes, we perform the following asymptotic expansion of θ_k:

$$\theta_k(x) = \theta_{k0}(x) + \theta_{k1}(x) + \theta_{k2}(x) + \cdots, \tag{47}$$

where for all $l \in \mathbb{N}$

$$\theta_{kl} = O(\epsilon^{l\gamma}), \qquad \text{and} \qquad \partial_x^\nu \theta_{kl}/\theta_{kl} = \theta_{kl}^{(\nu)}/\theta_{kl} = O(\epsilon^{-\nu\sigma}/a^\nu), \tag{48}$$

with $\gamma = 1 - \sigma$. Let us note that the high-order terms of the expansion (47) also contain the slow radial scale of length order a.

Remark 10. Let us notice that according to Ref. 77, it is equivalent to use the eikonal asymptotic expansion (47) and (48) or the two-scale asymptotic expansion of the envelope $\widehat{\phi}_{1\omega n}(x, y)$, where the second radial variable $y = \epsilon^{-\sigma}x$ should reproduce the intermediate scale of variation of length order $n^{-\sigma}a$ of the searched eigenmodes.

To clarify the idea, we shall consider $\gamma = 1/2,\ 1/3, \ldots$. We then have the asymptotic expansions in powers of the small parameter ϵ^γ for the other quantities (hereafter we include the relevant powers of ϵ^γ in the definition of the expansion terms)

$$\phi_{1\omega n} = \phi_{10\omega n} + \phi_{11\omega n} + \phi_{12\omega n} + \cdots, \tag{49}$$

$$w^\pm_{\mu b\omega n} = w^\pm_{0\mu b\omega n} + w^\pm_{1\mu b\omega n} + w^\pm_{2\mu b\omega n} + \cdots, \tag{50}$$

$$h^\pm_{\mu b\omega n} = h^\pm_{0\mu b\omega n} + h^\pm_{1\mu b\omega n} + h^\pm_{2\mu b\omega n} + \cdots, \tag{51}$$

$$\omega = \omega_0 + \omega_1 + \omega_2 + \cdots. \tag{52}$$

Now we have to obtain an asymptotic expansion of the differential operators $\mathcal{L}^\pm_{\mu b\omega n}$, $\mathcal{M}^\pm_{\mu b\omega n}$, and $Q_{\omega n}$. For this purpose, we dimensionalize them by using the length and time scales defined in Sec. II D. We then introduce dimensional variables, dimensional unknowns, and known quantities to make appear only the dimensional small parameters defined in Sec. II D. This straightforward but cumbersome and lengthy stage is omitted and we give directly the result. After gathering all the terms of same order by using the scale ordering of Sec. II D (particularly $\epsilon_\perp \simeq 1$ and $\epsilon_\omega \simeq \epsilon\epsilon_a$), we obtain for the operators $\mathcal{L}^\pm_{\mu b\omega n}$, $\mathcal{M}^\pm_{\mu b\omega n}$, and $Q_{\omega n}$ the following asymptotic expansions in powers of ϵ^γ:

$$\mathcal{L}^\pm_{\mu b\omega n} = \mathcal{L}^\pm_{0\mu b\omega n} + \mathcal{L}^\pm_{1\mu b\omega n} + \mathcal{L}^\pm_{2\mu b\omega n} + \cdots, \tag{53}$$

$$\mathcal{M}^\pm_{\mu b\omega n} = \mathcal{M}^\pm_{0\mu b\omega n} + \mathcal{M}^\pm_{1\mu b\omega n} + \mathcal{M}^\pm_{2\mu b\omega n} + \cdots, \tag{54}$$

$$Q_{\omega n} = Q_{0\omega n} + Q_{1\omega n} + Q_{2\omega n} + \ldots, \tag{55}$$

where the second-order terms are of order $\epsilon^{\min(1,2\gamma)}$. The operators involved in (53)-(55) are defined by

$$\mathcal{L}^{\pm}_{0\mu b\omega n}(\omega_0,\theta_{k0}) = -\mathrm{i}\omega_0 + \mathrm{i}n\left(\frac{\mu}{q_i} + \frac{a^{\pm}_{\mu b}}{\Omega_i}\right)\frac{b_\varphi}{R}\left(q'(\eta-\theta_{k0})\sin\eta + \mathfrak{q}\frac{\cos\eta}{r}\right) + \frac{b_\varphi}{qR}(\partial_\eta a^{\pm}_{\mu b} + a^{\pm}_{\mu b}\partial_\eta) - 2\frac{a^{\pm}_{\mu b}}{\Omega_i}\frac{b_\varphi}{R}\left(\sin\eta\,\partial_x a^{\pm}_{\mu b} - \frac{\cos\eta}{r}\partial_\eta a^{\pm}_{\mu b}\right), \quad (56)$$

$$\mathcal{L}^{\pm}_{1\mu b\omega n}(\omega_1,\theta_{k1}) = -\mathrm{i}nq'\left(\frac{\mu}{q_i} + \frac{a^{\pm}_{\mu b}}{\Omega_i}\right)\frac{b_\varphi}{R}\sin\eta\,\theta_{k1} - \mathrm{i}\omega_1 = (\theta_{k1}\partial_{\theta_{k0}} + \omega_1\partial_{\omega_0})\mathcal{L}^{\pm}_{0\mu b\omega n}, \quad (57)$$

$$\mathcal{L}^{\pm}_{2\mu b\omega n}(\omega_2,\theta_{k2}) = (\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0})\mathcal{L}^{\pm}_{0\mu b\omega n} - \left(\frac{\mu}{q_i} + \frac{a^{\pm}_{\mu b}}{\Omega_i}\right)\frac{b_\varphi}{R}\left(\frac{\cos\eta}{r}\partial_\eta + \sin\eta\,\partial_x\right) = (\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0})\mathcal{L}^{\pm}_{0\mu b\omega n} + \frac{\mathrm{i}}{n}\partial_{\mathfrak{q}}\mathcal{L}^{\pm}_{0\mu b\omega n}\partial_\eta - \frac{\mathrm{i}}{nq'}\partial_{\theta_{k0}}\mathcal{L}^{\pm}_{0\mu b\omega n}\partial_x, \quad (58)$$

$$\mathcal{M}^{\pm}_{0\mu b\omega n}(\theta_{k0}) = \mathrm{i}nb_\varphi\left\{\frac{1}{rB}\left(\mathfrak{q}\partial_x a^{\pm}_{\mu b} - q'(\eta-\theta_{k0})\partial_\eta a^{\pm}_{\mu b}\right) + \frac{q_i}{m_i}\frac{a^{\pm}_{\mu b}}{\Omega_i}\frac{1}{R}\left(q'(\eta-\theta_k)\sin\eta + \mathfrak{q}\frac{\cos\eta}{r}\right)\right\} + \frac{q_i}{m_i}\frac{b_\varphi}{qR}\partial_\eta, \quad (59)$$

$$\mathcal{M}^{\pm}_{1\mu b\omega n}(\theta_{k1}) = \mathrm{i}nq'\frac{b_\varphi}{B}\left(\frac{1}{r}\partial_\eta a^{\pm}_{\mu b} - \frac{\sin\eta}{R}a^{\pm}_{\mu b}\right)\theta_{k1} = (\theta_{k1}\partial_{\theta_{k0}} + \omega_1\partial_{\omega_0})\mathcal{M}^{\pm}_{0\mu b\omega n}, \quad (60)$$

$$\mathcal{M}^{\pm}_{2\mu b\omega n}(\theta_{k2}) = (\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0})\mathcal{M}^{\pm}_{0\mu b\omega n} - \frac{q_i}{m_i}\frac{a^{\pm}_{\mu b}}{\Omega_i}\frac{b_\varphi}{R}\left(\frac{\cos\eta}{r}\partial_\eta + \sin\eta\,\partial_x\right) + \frac{b_\varphi}{rB}\left(\partial_\eta a^{\pm}_{\mu b}\partial_x - \partial_x a^{\pm}_{\mu b}\partial_\eta\right) = (\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0})\mathcal{M}^{\pm}_{0\mu b\omega n} + \frac{\mathrm{i}}{n}\partial_{\mathfrak{q}}\mathcal{M}^{\pm}_{0\mu b\omega n}\partial_\eta - \frac{\mathrm{i}}{nq'}\partial_{\theta_{k0}}\mathcal{M}^{\pm}_{0\mu b\omega n}\partial_x, \quad (61)$$

$$Q_{0\omega n} = \frac{e\tau n_{i0}}{k_B T_{i0}} + \frac{n_{i0}}{\Omega_i B}\left(\frac{n^2 q^2}{r^2} + n^2 q'^2(\eta-\theta_{k0})^2\right), \quad (62)$$

$$Q_{1\omega n} = -2\frac{n_{i0}}{\Omega_i B}(nq')^2(\eta-\theta_{k0})\theta_{k1} - \mathrm{i}nq'\frac{n_{i0}}{\Omega_i B}\partial_x\theta_{k0} = \theta_{k1}\partial_{\theta_{k0}}Q_{0\omega n} - \frac{\mathrm{i}}{2}\frac{1}{nq'}\partial_x\theta_{k0}\partial^2_{\theta_{k0}}Q_{0\omega n}, \quad (63)$$

$$Q_{2\omega n} = -\mathrm{i}nq'\frac{n_{i0}}{\Omega_i B}\partial_x\theta_{k1} + \frac{n_{i0}}{\Omega_i B}(nq')^2\left(\theta^2_{k1} - 2(\eta-\theta_{k0})\theta_{k2}\right) + \mathrm{i}n(\eta-\theta_{k0})\frac{1}{rR}\partial_x\left(rRq'\frac{n_{i0}}{\Omega_i B}\ \cdot\ \right) + \mathrm{i}nq\frac{1}{r^2R}\partial_\eta\left(R\frac{n_{i0}}{\Omega_i B}\ \cdot\ \right)$$

$$= \theta_{k2}\partial_{\theta_{k0}}Q_{0\omega n} - \left(-\frac{1}{2}\theta_{k1}^2 + \frac{i}{2}\frac{1}{nq'}\partial_x\theta_{k1}\right)\partial^2_{\theta_{k0}}Q_{0\omega n}$$
$$+ \mathrm{i}n(\eta-\theta_{k0})\frac{1}{rR}\partial_x\left(rRq'\frac{n_{i0}}{\Omega_i B}\ \cdot\ \right) + \mathrm{i}nq\frac{1}{r^2R}\partial_\eta\left(R\frac{n_{i0}}{\Omega_i B}\ \cdot\ \right). \tag{64}$$

Let us now make the following important remark.

Remark 11. In (58) and (61) terms that are at least of second order in $\epsilon^{2\gamma}$ fall into two groups: if they involve radial or poloidal derivatives, they are of order ϵ; if they do not, they are of order $\epsilon^{2\gamma}$. Upon choosing $\gamma = 1/2$ all the terms must be taken into account in defining the second-order group. On the contrary, for $\gamma < 1/2$ only the terms without poloidal or radial derivatives are needed, so that partial derivatives with respect to the variables x and η drop out. The same applies to (64).

Substituting the asymptotic expansions (49) and (51) and (53)-(55) into (45)-(46) with $\mathfrak{J}_\mu = 1$, we get, at the zeroth order, the system

$$\mathcal{L}^{\pm}_{0\mu b\omega n}\left(\frac{h^{\pm}_{0\mu b\omega n}}{a^{\pm}_{\mu b}}\right) + \mathcal{M}^{\pm}_{0\mu b\omega n}\phi_{10\omega n} - \frac{q_i}{m_i}\mathcal{L}^{\pm}_{0\mu b\omega n}\left(\frac{\phi_{10\omega n}}{a^{\pm}_{\mu b}}\right) = 0, \tag{65}$$

$$\left\{Q_{0\omega n} + 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left(\frac{1}{a^{+}_{\mu b}} - \frac{1}{a^{-}_{\mu b}}\right)\right\}\phi_{10\omega n} = 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left(\frac{h^{+}_{0\mu b\omega n}}{a^{+}_{\mu b}} - \frac{h^{-}_{0\mu b\omega n}}{a^{-}_{\mu b}}\right), \tag{66}$$

at the first order,

$$\mathcal{L}^{\pm}_{0\mu b\omega n}\left(\frac{h^{\pm}_{1\mu b\omega n}}{a^{\pm}_{\mu b}}\right) + \mathcal{M}^{\pm}_{0\mu b\omega n}\phi_{11\omega n} - \frac{q_i}{m_i}\mathcal{L}^{\pm}_{0\mu b\omega n}\left(\frac{\phi_{11\omega n}}{a^{\pm}_{\mu b}}\right)$$
$$+ \mathcal{L}^{\pm}_{1\mu b\omega n}\left(\frac{h^{\pm}_{0\mu b\omega n}}{a^{\pm}_{\mu b}}\right) + \mathcal{M}^{\pm}_{1\mu b\omega n}\phi_{10\omega n} - \frac{q_i}{m_i}\mathcal{L}^{\pm}_{1\mu b\omega n}\left(\frac{\phi_{10\omega n}}{a^{\pm}_{\mu b}}\right) = 0, \tag{67}$$

$$\left\{Q_{0\omega n} + 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left(\frac{1}{a^{+}_{\mu b}} - \frac{1}{a^{-}_{\mu b}}\right)\right\}\phi_{11\omega n}$$
$$+ Q_{1\omega n}\phi_{10\omega n} = 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left(\frac{h^{+}_{1\mu b\omega n}}{a^{+}_{\mu b}} - \frac{h^{-}_{1\mu b\omega n}}{a^{-}_{\mu b}}\right), \tag{68}$$

and at the second order,

$$\mathcal{L}^{\pm}_{0\mu b\omega n}\left(\frac{h^{\pm}_{2\mu b\omega n}}{a^{\pm}_{\mu b}}\right) + \mathcal{M}^{\pm}_{0\mu b\omega n}\phi_{12\omega n} - \frac{q_i}{m_i}\mathcal{L}^{\pm}_{0\mu b\omega n}\left(\frac{\phi_{12\omega n}}{a^{\pm}_{\mu b}}\right) + \mathcal{L}^{\pm}_{1\mu b\omega n}\left(\frac{h^{\pm}_{1\mu b\omega n}}{a^{\pm}_{\mu b}}\right) + \mathcal{M}^{\pm}_{1\mu b\omega n}\phi_{11\omega n}$$
$$- \frac{q_i}{m_i}\mathcal{L}^{\pm}_{1\mu b\omega n}\left(\frac{\phi_{11\omega n}}{a^{\pm}_{\mu b}}\right) + \mathcal{L}^{\pm}_{2\mu b\omega n}\left(\frac{h^{\pm}_{0\mu b\omega n}}{a^{\pm}_{\mu b}}\right) + \mathcal{M}^{\pm}_{2\mu b\omega n}\phi_{10\omega n} - \frac{q_i}{m_i}\mathcal{L}^{\pm}_{2\mu b\omega n}\left(\frac{\phi_{10\omega n}}{a^{\pm}_{\mu b}}\right) = 0, \tag{69}$$

$$\left\{Q_{0\omega n} + 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left(\frac{1}{a^{+}_{\mu b}} - \frac{1}{a^{-}_{\mu b}}\right)\right\}\phi_{12\omega n} + Q_{1\omega n}\phi_{11\omega n} + Q_{2\omega n}\phi_{10\omega n}$$
$$= 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left(\frac{h^{+}_{2\mu b\omega n}}{a^{+}_{\mu b}} - \frac{h^{-}_{2\mu b\omega n}}{a^{-}_{\mu b}}\right). \tag{70}$$

Before solving successively the three previous systems, we define the following new quantities:

$$\omega^{\pm}_{d\mu b} = n\left(\frac{\mu}{q_i} + \frac{a^{\pm 2}_{\mu b}}{\Omega_i}\right)\frac{b_\varphi}{R}\left(q'(\eta-\theta_{k0})\sin\eta + \mathfrak{q}\frac{\cos\eta}{r}\right),$$

$$\omega^{\pm}_{*\mu b} = \frac{T_{e0}}{e}\frac{n}{B}\frac{b_\varphi}{a^{\pm}_{\mu b}}\left\{\frac{1}{r}\left(\mathfrak{q}\partial_x a^{\pm}_{\mu b} - q'(\eta-\theta_{k0})\partial_\eta a^{\pm}_{\mu b}\right) + \frac{a^{\pm}_{\mu b}}{R}\left(q'(\eta-\theta_k)\sin\eta + \mathfrak{q}\frac{\cos\eta}{r}\right)\right\},$$

$$\omega^{\pm}_{\diamond\mu b} = -2\frac{a^{\pm}_{\mu b}}{\Omega_i}\frac{b_\varphi}{R}\left(\sin\eta\,\partial_x a^{\pm}_{\mu b} - \frac{\cos\eta}{r}\partial_\eta a^{\pm}_{\mu b}\right),$$

$$\Omega^{\pm}_{*\mu b} = \omega^{\pm}_{*\mu b} + \frac{q_i}{m_i}\frac{T_{e0}}{e}\frac{1}{|a^{\pm}_{\mu b}|^2}(\omega_0 - \omega^{\pm}_{d\mu b} + \mathrm{i}\omega^{\pm}_{\diamond\mu b}).$$

In the case of symmetric equilibrium contours, i.e., $a^{\pm}_{\mu b}(r,\theta) = \pm a_{\mu b}(r,\theta)$, the previous definitions simplify: $\omega^{\pm}_{d\mu b} = \omega_{d\mu b}$, $\omega^{\pm}_{*\mu b} = \omega_{*\mu b}$, $\omega^{\pm}_{\diamond\mu b} = \omega_{\diamond\mu b}$, and $\Omega^{\pm}_{*\mu b} = \Omega_{*\mu b}$.

B. The zeroth-order system

In this section we recast the system (65) and (66) as an integral equation for the potential $\phi_{10\omega n}$, given by the following proposition.

Proposition 2. The zeroth-order system (65) and (66) is equivalent to the integral equation

$$\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n} = 0, \tag{71}$$

where

$$\mathcal{L}^{\circ}_{\mathscr{C}\omega n} = Q^{\circ}_{\omega n} + \mathcal{L}^{\circ}_{O\omega n} + \mathcal{L}^{\circ}_{C\omega n}.$$

Here, the operators $Q^{\circ}_{\omega n}$, $\mathcal{L}^{\circ}_{O\omega n}$, and $\mathcal{L}^{\circ}_{C\omega n}$ are defined by

$$Q^{\circ}_{\omega n} = Q_{0\omega n} + 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left(\frac{1}{a^{+}_{\mu b}} - \frac{1}{a^{-}_{\mu b}}\right), \qquad \mathcal{L}^{\circ}_{O\omega n} = \sum_{\mu b\in O}\mathcal{L}^{\circ}_{O\mu b\omega n},$$

$$\mathcal{L}^{\circ}_{C\omega n} = \sum_{\mu b\in C}\mathcal{L}^{\circ}_{C\mu b\omega n},$$

$$\mathcal{L}^{\circ}_{O\mu b\omega n}\psi = \mathrm{i}2\pi\frac{\Omega_i}{q_i}\frac{\mathcal{A}_{\mu b}}{a_{\mu b}}\int_{-\infty}^{+\infty} d\widehat{\eta}\left(\frac{e}{T_{e0}}\frac{qR}{b_\varphi}\Omega_{*\mu b}a_{\mu b}\psi\right)(\widehat{\eta})\exp\left(-\mathrm{i}\,\mathrm{sign}(\eta-\widehat{\eta})\mathcal{I}_{\mu b}(\eta,\widehat{\eta})\right),$$

$$\begin{aligned}\mathcal{L}^{\circ}_{C\mu b\omega n}\psi = \sum_{\ell\in\mathbb{Z}} -2\pi\frac{\Omega_i}{q_i}\frac{\mathcal{A}_{\mu b}}{a_{\mu b}}&\frac{2\mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}]}(\eta)}{\sin\mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b})}\\ &\left\{\int_{\eta}^{\theta^{+,\ell}_{L\mu b}} d\widehat{\eta}\left(\frac{e}{T_{e0}}\frac{qR}{b_\varphi}\Omega_{*\mu b}a_{\mu b}\psi\right)(\widehat{\eta})\cos\mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\widehat{\eta})\cos\mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\eta)\right.\\ &\left.+\int_{\theta^{-,\ell}_{L\mu b}}^{\eta} d\widehat{\eta}\left(\frac{e}{T_{e0}}\frac{qR}{b_\varphi}\Omega_{*\mu b}a_{\mu b}\psi\right)(\widehat{\eta})\cos\mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\widehat{\eta})\cos\mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\eta)\right\},\end{aligned}$$

where

$$\mathcal{I}_{\mu b}(\eta,\widehat{\eta}) = \int_{\eta}^{\widehat{\eta}} d\widetilde{\eta}\,\frac{qR}{b_\varphi a_{\mu b}}(\omega_0 - \omega_{d\mu b} + \mathrm{i}\omega_{\diamond\mu b})$$

and $\theta^{\pm,\ell}_{L\mu b}(r) = \pm\theta_{L\mu b}(r) + 2\pi\ell$.

Proof. We distinguish the case of open contours (the set O) and closed contours (the set C). Let us first deal with the set O. Integration of Equation (65) in η-variable gives

$$\begin{aligned}h^{\pm}_{0\mu b\omega n}(\eta) &= h^{\pm}_{0\mu b\omega n}(\eta_0)\exp\left(\mathrm{i}\int_{\eta_0}^{\eta} d\widetilde{\eta}\,\frac{qR}{b_\varphi a^{\pm}_{\mu b}}(\omega_0 - \omega^{\pm}_{d\mu b} + \mathrm{i}\omega^{\pm}_{\diamond\mu b})\right)\\ &\quad - \mathrm{i}\int_{\eta_0}^{\eta} d\widehat{\eta}\left(\frac{e}{T_{e0}}\frac{qR}{b_\varphi}\Omega^{\pm}_{*\mu b}a^{\pm}_{\mu b}\phi_{10\omega n}\right)(\widehat{\eta})\exp\left(\mathrm{i}\int_{\widehat{\eta}}^{\eta} d\widetilde{\eta}\,\frac{qR}{b_\varphi a^{\pm}_{\mu b}}(\omega_0 - \omega^{\pm}_{d\mu b} + \mathrm{i}\omega^{\pm}_{\diamond\mu b})\right).\end{aligned} \tag{72}$$

Since here we consider open contours and $\eta \in \mathbb{R}$, it is natural to take the following boundary conditions: $w^{\pm}_{0\mu b\omega n}(\pm\infty) = 0$ for the contours and $\phi_{10\omega n}(\pm\infty) = 0$ for the potential. As a consequence, we get $h^{\pm}_{0\mu b\omega n}(\pm\infty) = 0$ for all $(\mu, b) \in \mathcal{O}$. Moreover from Sec. III B, we have seen that if we assume that the "initial" or boundary condition $a^{\pm}_{\mu b}(r,0)$ is symmetric — so that $a^{\pm}_{\mu b}(r,0) = \pm a_{\mu b}(r,0)$, with $a_{\mu b}(r,0) \geq 0$ — then the contours $a^{\pm}_{\mu b}(r,\theta)$ also satisfy this property. Therefore we get $a^{\pm}_{\mu b}(r,\theta) = \pm a_{\mu b}(r,\theta)$, $\omega^{\pm}_{d\mu b} = \omega_{d\mu b}$, $\omega^{\pm}_{*\mu b} = \omega_{*\mu b}$, $\omega^{\pm}_{\diamond\mu b} = \omega_{\diamond\mu b}$, and $\Omega^{\pm}_{*\mu b} = \Omega_{*\mu b}$. Consequently, for all $(\mu, b) \in \mathcal{O}$, we obtain

$$h^{\pm}_{0\mu b\omega n}(\eta) = \mp \mathrm{i} \int_{\mp\infty}^{\eta} d\widehat{\eta} \left(\frac{e}{T_{e0}} \frac{qR}{b_\varphi} \Omega_{*\mu b} a_{\mu b} \phi_{10\omega n} \right)(\widehat{\eta}) \exp\left(-\mathrm{i}\,\mathrm{sign}(\eta - \widehat{\eta}) \mathcal{I}_{\mu b}(\eta, \widehat{\eta})\right), \tag{73}$$

where

$$\mathcal{I}_{\mu b}(\eta, \widehat{\eta}) = \int_{\eta}^{\widehat{\eta}} d\widetilde{\eta}\, \frac{qR}{b_\varphi a_{\mu b}} (\omega_0 - \omega_{d\mu b} + \mathrm{i}\omega_{\diamond\mu b}).$$

Let us now deal with closed contours. As it has been shown for equilibrium contours $\{a^{\pm}_{\mu b}\}_{(\mu,b)\in\mathcal{C}}$ (see Sec. III B), it is natural to assume that for all $(\mu, b) \in \mathcal{C}$, the perturbation contours $w^{+}_{0\mu b\omega n}$ and $w^{-}_{0\mu b\omega n}$ close by meeting to each other at two angles $\theta^1_{L\mu b}(r)$ and $\theta^2_{L\mu b}(r)$ where they vanish, and thus form a multi-valued (double-valued function) closed contour. Even if it is not necessary, we can assume that the limit angles $\theta^1_{L\mu b}(r)$ and $\theta^2_{L\mu b}(r)$ of the perturbation contours are the same as those of the corresponding equilibrium contours, i.e., $\theta^2_{L\mu b}(r) = \theta^{+,\ell}_{L\mu b}(r) = \theta_{L\mu b}(r) + 2\pi\ell$, and $\theta^1_{L\mu b}(r) = \theta^{-,\ell}_{L\mu b}(r) = -\theta_{L\mu b}(r) + 2\pi\ell$, with $\ell \in \mathbb{Z}$, $\theta_{L\mu b}(r) := |\theta^{\pm}_{L\mu b}(r)|$ and $\theta^{\pm}_{L\mu b}(r)$ given by (30). In order that the contours connect each other, the boundary conditions for the contours belonging to the set $\mathcal{C}$, should be $w^{+}_{0\mu b\omega n}(\theta^1_{L\mu b}) = w^{-}_{0\mu b\omega n}(\theta^1_{L\mu b})$ and $w^{+}_{0\mu b\omega n}(\theta^2_{L\mu b}) = w^{-}_{0\mu b\omega n}(\theta^2_{L\mu b})$. Assuming now that $\phi_{10\omega n}$ is continuous, we obtain the boundary conditions

$$h^{+}_{0\mu b\omega n}(\theta^1_{L\mu b}) = h^{-}_{0\mu b\omega n}(\theta^1_{L\mu b}), \quad \text{and} \quad h^{+}_{0\mu b\omega n}(\theta^2_{L\mu b}) = h^{-}_{0\mu b\omega n}(\theta^2_{L\mu b}), \quad \forall(\mu, b) \in \mathcal{C}.$$

By taking $(\eta_0 = \theta^1_{L\mu b}, \eta = \theta^2_{L\mu b})$ for $h^{-}_{0\mu b\omega n}$ defined by (72), $(\eta_0 = \theta^2_{L\mu b}, \eta = \theta^1_{L\mu b})$ for $h^{+}_{0\mu b\omega n}$ defined by (72), and using the previous boundary conditions we obtain the 2×2 linear system

$$\begin{aligned} &h^{+}_{0\mu b\omega n}(\theta^k_{L\mu b}) - h^{+}_{0\mu b\omega n}(\theta^j_{L\mu b}) \exp\left((-1)^j \mathcal{I}_{\mu b}(\theta^j_{L\mu b}, \theta^k_{L\mu b})\right) \\ &= (-1)^k \mathrm{i} \int_{\theta^j_{L\mu b}}^{\theta^k_{L\mu b}} d\eta \left(\frac{e}{T_{e0}} \frac{qR}{b_\varphi} \Omega_{*\mu b} a_{\mu b} \phi_{10\omega n} \right)(\eta) \exp\left((-1)^k \mathrm{i} \left(\mathcal{I}_{\mu b}(\theta^j_{L\mu b}, \eta) - \mathcal{I}_{\mu b}(\theta^j_{L\mu b}, \theta^k_{L\mu b})\right)\right), \end{aligned}$$

with $(j, k) \in \{(1, 2), (2, 1)\}$. Solving the previous linear system gives

$$\begin{aligned} h_{0\mu b\omega n}(\theta^1_{L\mu b}) &:= h^{\pm}_{0\mu b\omega n}(\theta^1_{L\mu b}) \\ &= \sin^{-1} \mathcal{I}_{\mu b}(\theta^1_{L\mu b}, \theta^2_{L\mu b}) \int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\eta \left(\frac{e}{T_{e0}} \frac{qR}{b_\varphi} \Omega_{*\mu b} a_{\mu b} \phi_{10\omega n} \right)(\eta) \cos \mathcal{I}_{\mu b}(\theta^2_{L\mu b}, \eta), \end{aligned} \tag{74}$$

which finally leads to

$$\begin{aligned} h^{\pm}_{0\mu b\omega n}(\eta) = \exp\left(\pm \mathrm{i} \mathcal{I}_{\mu b}(\theta^1_{L\mu b}, \eta)\right) &\left\{ \mp \mathrm{i} \int_{\theta^1_{L\mu b}}^{\eta} d\widehat{\eta} \left(\frac{e}{T_{e0}} \frac{qR}{b_\varphi} \Omega_{*\mu b} a_{\mu b} \phi_{10\omega n} \right)(\widehat{\eta}) \exp\left(\mp \mathrm{i} \mathcal{I}_{\mu b}(\theta^1_{L\mu b}, \widehat{\eta})\right) \right. \\ &\left. + \sin^{-1} \mathcal{I}_{\mu b}(\theta^1_{L\mu b}, \theta^2_{L\mu b}) \int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\eta \left(\frac{e}{T_{e0}} \frac{qR}{b_\varphi} \Omega_{*\mu b} a_{\mu b} \phi_{10\omega n} \right)(\eta) \cos \mathcal{I}_{\mu b}(\theta^2_{L\mu b}, \eta) \right\}. \end{aligned} \tag{75}$$

Substituting (73) and (75) into the quasi-neutrality equation (66), we obtain Proposition 2. □

Remark 12. Further, we will discuss the question of the well-posedness of the integral Equation (71). Let us note that this one-dimensional integral equation depends parametrically on the radius through the x-variable. The solution of the integral Equation (71) gives the geometric structure of the eigenmode in the poloidal direction or along a magnetic field line, locally in radius.

In the integral Equation (71) the map $[x_{\min}, x_{\max}] \ni x \longmapsto \theta_{k0}(x) \in \mathbb{R}$ remains unknown. It will actually be determined by solving the first-order problem (67) and (68), a matter handled in Sec. IV C. The angle $\theta_{k0}(x)$, called the ballooning angle, represents for each radius, the angle at which is centered the poloidal envelope of the eigenmode $\phi_{10\omega n}$. We will see that the ballooning angle can be taken independent of the radius.

C. The first-order system

In this section we solve the system (67) and (68) and determine the map $[x_{\min}, x_{\max}] \ni x \longmapsto \theta_{k0}(x) \in \mathbb{R}$. We first recast the system (67) and (68) in a suitable form given by the following proposition.

Proposition 3. The system (67) and (68) is equivalent to

$$\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{11\omega n} + \left(\theta_{k1}\partial_{\theta_{k0}}\mathcal{L}^{\circ}_{\mathscr{C}\omega n} + \omega_1\partial_{\omega_0}\mathcal{L}^{\circ}_{\mathscr{C}\omega n} - \frac{\mathrm{i}}{2}\frac{1}{nq'}\partial_x\theta_{k0}\partial^2_{\theta_{k0}}\mathcal{Q}_{0\omega n}\right)\phi_{10\omega n} = 0. \tag{76}$$

Proof. Let us first deal with the set $\mathcal{O}$ of open contours. For this, let us introduce the following definitions:

$$\alpha_{\mu b} = -\mathrm{i}\left(\theta_{k1}\partial_{\theta_{k0}} + \omega_1\partial_{\omega_0}\right)\mathcal{L}_{0\mu b\omega n} = -nq'\left(\frac{\mu}{q_i} + \frac{a^2_{\mu b}}{\Omega_i}\right)\frac{b_\varphi}{R}\theta_{k1}\sin\eta - \omega_1,$$

$$\beta_{\mu b} = -\mathrm{i}\left(\theta_{k1}\partial_{\theta_{k0}} + \omega_1\partial_{\omega_0}\right)\mathcal{M}_{0\mu b\omega n} = nq'\frac{b_\varphi}{B}\left(\frac{1}{r}\partial_\eta a_{\mu b} - \frac{\sin\eta}{R}a_{\mu b}\right)\theta_{k1},$$

$$\gamma_{\mu b} = \frac{e}{T_{e0}}\frac{qR}{b_\varphi}\Omega_{*\mu b}a_{\mu b},$$

$$\mathcal{K}_{\mu b}(\eta,\widehat{\eta}) = \exp\left(-\mathrm{i}\,\mathrm{sign}(\eta-\widehat{\eta})\mathcal{I}_{\mu b}(\eta,\widehat{\eta})\right).$$

Following the same ideas for dealing with the zeroth-order system, integration of (67) with respect to η-variable gives, for open contours, i.e., for all $(\mu, b) \in \mathcal{O}$,

$$h^{\pm}_{1\mu b\omega n}(\eta) = -\mathrm{i}\int_{\mp\infty}^{\eta} d\widehat{\eta}\left(\pm\gamma_{\mu b}\phi_{11\omega n} + \frac{qR}{b_\varphi}\left[\pm\frac{\alpha_{\mu b}}{a_{\mu b}}h^{\pm}_{0\mu b\omega n} + \left\{\pm\beta_{\mu b} \mp \frac{q_i}{m_i}\frac{\alpha_{\mu b}}{a_{\mu b}}\right\}\phi_{10\omega n}\right]\right)(\widehat{\eta})\,\mathcal{K}_{\mu b}(\eta,\widehat{\eta}). \tag{77}$$

Using (73), (77) and Fubini's theorem to permutate the order of integrations, we obtain

$$(h^{+}_{1\mu b\omega n} + h^{-}_{1\mu b\omega n})(\eta) = -\mathrm{i}\left\{\int_{-\infty}^{+\infty} d\widehat{\eta}\left(\gamma_{\mu b}\phi_{11\omega n} + \frac{qR}{b_\varphi}\left[\beta_{\mu b} - \frac{q_i}{m_i}\frac{\alpha_{\mu b}}{a_{\mu b}}\right]\phi_{10\omega n}\right)(\widehat{\eta})\,\mathcal{K}_{\mu b}(\eta,\widehat{\eta}) -\mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,(\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\,\mathrm{sign}(\eta-\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\int_{\widehat{\eta}}^{\eta} d\widetilde{\eta}\left(\frac{qR}{b_\varphi}\frac{\alpha_{\mu b}}{a_{\mu b}}\right)(\widetilde{\eta})\right\}. \tag{78}$$

Let us now deal with closed contours. Using (74), we obtain

$$h_{1\mu b\omega n}(\theta^1_{L\mu b}) := h^{\pm}_{1\mu b\omega n}(\theta^1_{L\mu b}) = \sin^{-1}\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\theta^2_{L\mu b})\left[\int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\eta\,(\gamma_{\mu b}\phi_{11\omega n})(\eta)\cos\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\eta) + \int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\eta\left(\frac{qR}{b_\varphi}\left[\beta_{\mu b} - \frac{q_i}{m_i}\frac{\alpha_{\mu b}}{a_{\mu b}}\right]\phi_{10\omega n}\right)(\eta)\cos\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\eta) + \int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\widehat{\eta}\left(\frac{qR}{b_\varphi}\frac{\alpha_{\mu b}}{a_{\mu b}}\right)(\widehat{\eta})\left\{\frac{\cos\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\theta^2_{L\mu b})}{\sin\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\theta^2_{L\mu b})}\int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\widetilde{\eta}\,(\gamma_{\mu b}\phi_{10\omega n})(\widetilde{\eta})\cos\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\widetilde{\eta}) - \int_{\theta^1_{L\mu b}}^{\widehat{\eta}} d\widetilde{\eta}\,(\gamma_{\mu b}\phi_{10\omega n})(\widetilde{\eta})\sin\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\widetilde{\eta})\right\}\right], \tag{79}$$

which leads to

$$h^{\pm}_{1\mu b\omega n}(\eta) = h^{\pm}_{0\mu b\omega n}(\theta^1_{L\mu b})\exp\left(\pm \mathrm{i}\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\eta)\right) - \mathrm{i}\int_{\theta^1_{L\mu b}}^{\eta} d\widetilde{\eta}\,\Big(\pm\gamma_{\mu b}\phi_{11\omega n} + \frac{qR}{b_\varphi}\left\{\pm\frac{\alpha_{\mu b}}{a_{\mu b}}h^{\pm}_{0\mu b\omega n} \pm \left[\beta_{\mu b} - \frac{q_i}{m_i}\frac{\alpha_{\mu b}}{a_{\mu b}}\right]\phi_{10\omega n}\right\}\Big)(\widetilde{\eta})\exp\left(\mp \mathrm{i}\mathcal{I}_{\mu b}(\eta,\widetilde{\eta})\right). \quad (80)$$

Using (74) and (75) and (79) and (80), we get

$$\begin{aligned}(h^{+}_{1\mu b\omega n} + h^{-}_{1\mu b\omega n})(\eta) = 2\Big[&h_{1\mu b\omega n}(\theta^1_{L\mu b})\cos\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\eta) \\ &- \int_{\theta^1_{L\mu b}}^{\eta} d\widetilde{\eta}\left(\gamma_{\mu b}\phi_{11\omega n} + \left[\beta_{\mu b} - \frac{q_i}{m_i}\frac{\alpha_{\mu b}}{a_{\mu b}}\right]\phi_{10\omega n}\right)(\widetilde{\eta})\sin\mathcal{I}_{\mu b}(\eta,\widetilde{\eta}) \\ &+ h_{0\mu b\omega n}(\theta^1_{L\mu b})\int_{\theta^1_{L\mu b}}^{\eta} d\widetilde{\eta}\left(\frac{qR}{b_\varphi}\frac{\alpha_{\mu b}}{a_{\mu b}}\right)(\widetilde{\eta})\sin\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\widetilde{\eta}) \\ &- \int_{\theta^1_{L\mu b}}^{\eta} d\widetilde{\eta}\left(\frac{qR}{b_\varphi}\frac{\alpha_{\mu b}}{a_{\mu b}}\right)(\widetilde{\eta})\int_{\theta^1_{L\mu b}}^{\widetilde{\eta}} d\widehat{\eta}\,(\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\cos\mathcal{I}_{\mu b}(\eta,\widehat{\eta})\Big]. \end{aligned} \quad (81)$$

Substituting expressions (78) and (81) into the quasi-neutrality equation (68), we obtain the desired integral equation. We now recast this integral equation in a more compact form by observing the following identities:

$$\frac{qR}{b_\varphi}\frac{\alpha_{\mu b}}{a_{\mu b}} = \left(\theta_{k1}\partial_{\theta_{k0}} + \omega_1\partial_{\omega_0}\right)\partial_\eta\mathcal{I}_{\mu b}(\eta,\eta_0), \quad \forall\eta_0\in\mathbb{R}, \quad (82)$$

$$\beta_{\mu b} - \frac{q_i}{m_i}\frac{\alpha_{\mu b}}{a_{\mu b}} = \frac{b_\varphi}{qR}\left(\theta_{k1}\partial_{\theta_{k0}} + \omega_1\partial_{\omega_0}\right)\gamma_{\mu b}. \quad (83)$$

Using (82) and (83), the right hand side of (78) can be recast as

$$-2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in O}\frac{\mathcal{A}_{\mu b}}{a_{\mu b}}(h^{+}_{1\mu b\omega n} + h^{-}_{1\mu b\omega n}) = \left(\theta_{k1}\partial_{\theta_{k0}}\mathcal{L}^{\circ}_{O\omega n} + \omega_1\partial_{\omega_0}\mathcal{L}^{\circ}_{O\omega n}\right)\phi_{10\omega n}. \quad (84)$$

Using Fubini's theorem to permutate the order of integrations, we get

$$\begin{aligned}-\int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\widetilde{\eta}\,\left(\theta_{k1}\partial_{\theta_{k0}} + \omega_1\partial_{\omega_0}\right)\partial_\eta\mathcal{I}_{\mu b}(\widetilde{\eta},\eta_0)\int_{\theta^1_{L\mu b}}^{\widetilde{\eta}} d\widehat{\eta}\,(\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\sin\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\widehat{\eta}) \\ = \int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\eta\,(\gamma_{\mu b}\phi_{10\omega n})(\eta)\left(\theta_{k1}\partial_{\theta_{k0}} + \omega_1\partial_{\omega_0}\right)\cos\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\eta),\end{aligned}$$

so that (82) and (83) and the previous formula allow us to recast the right hand side of (81) as

$$-2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in C}\frac{\mathcal{A}_{\mu b}}{a_{\mu b}}(h^{+}_{1\mu b\omega n} + h^{-}_{1\mu b\omega n}) = \left(\theta_{k1}\partial_{\theta_{k0}}\mathcal{L}^{\circ}_{C\omega n} + \omega_1\partial_{\omega_0}\mathcal{L}^{\circ}_{C\omega n}\right)\phi_{10\omega n}.$$

Finally, using (63) and (84) and the previous equation, the system (67) and (68) can be rewritten as (76) of Proposition 3. □

The operator $\mathcal{L}^{\circ}_{\mathscr{C}\omega n}$ is non-self-adjoint. We denote by $\mathcal{L}^{\circ\star}_{\mathscr{C}\omega n}$ the dual operator of $\mathcal{L}^{\circ}_{\mathscr{C}\omega n}$, determined through the Hermitian scalar product in $L^2(\mathbb{R}_\eta)$, i.e., $\langle f,g\rangle_{L^2} = \int_{\mathbb{R}} f(\eta)g^\star(\eta)d\eta$ with $(\cdot)^\star$ the transposed conjugate. More precisely, we have

$$\langle\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\varphi,\psi\rangle_{L^2} = \langle\varphi,\mathcal{L}^{\circ\star}_{\mathscr{C}\omega n}\psi\rangle_{L^2}, \quad \forall\varphi,\psi\in L^2(\mathbb{R}_\eta).$$

We then define $\widehat{\phi}_{10\omega n}$ as the solution of the equation $\mathcal{L}^{\circ\star}_{\mathscr{C}\omega n}\widehat{\phi}_{10\omega n} = 0$. Let us now consider the equation

$$\mathscr{H}(\omega_0,\theta_{k0}(x),x) = \left\langle\widehat{\phi}_{10\omega n},\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n}\right\rangle_{L^2} = 0. \quad (85)$$

The function of two variables $\mathcal{H}_{\omega_0}(\theta_{k0}, x)$, defined for $x \in [x_{\min}, x_{\max}] = [r_{\min} - r_0, r_{\max} - r_0]$, $\theta_{k0} \in \mathbb{R}/2\pi\mathbb{Z}$ and $\omega_0 \in \mathbb{C}$, can be viewed as a Hamiltonian in the phase-space (θ_{k0}, x), which depends parametrically on ω_0 and where the variables (θ_{k0}, x) are conjugate variables. The equation for the characteristic curves in the two-dimensional (θ_{k0}, x)-phase-space is given by the Hamilton equations

$$\frac{d\theta_{k0}}{d\tau} = \frac{\partial \mathcal{H}_{\omega_0}}{\partial x}, \quad \frac{dx}{d\tau} = -\frac{\partial \mathcal{H}_{\omega_0}}{\partial \theta_{k0}}, \quad \left(\frac{d\mathcal{H}_{\omega_0}}{d\tau} = 0\right), \tag{86}$$

where $x \in [x_{\min}, x_{\max}]$, $\theta_{k0} \in \mathbb{R}/2\pi\mathbb{Z}$, $\omega_0 \in \mathbb{C}$, and $\tau \in \mathbb{R}_+$.

For such a one-dimensional autonomous Hamiltonian, i.e., with one degree of freedom, we know that the system is integrable. It is clear that, qualitatively, the determination of the function $\theta_{k0}(x)$ will depend on the topology of the phase portrait associated to this dynamical system. The numerical studies reported in the companion paper[24] indicate that, typically, the phase portrait has the same topology as the standard nonlinear pendulum, usually called cat's eyes (see Fig. 4). It is of interest to find general conditions ensuring such a topology. This is the goal of the following proposition. We have used p_0 to denote a generic radial profile, which can be, for example, the density n_{i0}, one of the temperatures T_{i0} and T_{e0}, the radial profiles $a^{\circ}_{\mu b}$ of the contours, or the safety factor q.

Proposition 4. Let p_0 denote a radial profile in $\mathcal{C}^2_b([r_{\min}, r_{\max}])$. Suppose that

(i) *there exists a unique point $x_c \in]x_{\min}, x_{\max}[$ such that*

$$(\partial_x \mathcal{H}_{\omega_0})(\theta_{k0}, x_c) = 0, \quad \forall \theta_{k0} \in \mathbb{R}/2\pi\mathbb{Z},$$

(ii) *at the critical points of $\mathcal{H}_{\omega_0}$ where $\partial^2_{\theta_{k0}} \mathcal{H}_{\omega_0}$ and $\partial^2_x \mathcal{H}_{\omega_0}$ have the same sign, the following stability condition holds:*

$$\left|\partial^2_{\theta_{k0}, x} \mathcal{H}_{\omega_0}\right|^2 < \left|\partial^2_{\theta_{k0}} \mathcal{H}_{\omega_0}\right| \left|\partial^2_x \mathcal{H}_{\omega_0}\right|.$$

Then, the topology of the integral curves of the Hamiltonian vector field $(\partial_x \mathcal{H}_{\omega_0}, -\partial_{\theta_{k0}} \mathcal{H}_{\omega_0})^T$ in phase-space are those of the classical nonlinear pendulum. In other words the phase-space contains an alternating sequence of X-points (saddle hyperbolic fixed points) and O-points (center or elliptic fixed points) along the line $x = x_c$. Moreover, the characteristic curves in phase-space are periodic and of two different topologies: the first one, called rotation, corresponds to open trajectories (passing orbits), while the second one, called oscillation (or sometimes vibration or libration), corresponds to closed trajectories (trapped orbits). The periodic separatrix curves connect the various X-points and separate rotations from oscillations.

Proof. First, using the regularity assumptions of Proposition 4, we have $\mathcal{H}_{\omega_0}(\theta_{k0}, x) \in \mathcal{C}^2_b((\mathbb{R}/2\pi\mathbb{Z}) \times [x_{\min}, x_{\max}])$. Therefore, using the Cauchy–Lipschitz–Picard theorem, we obtain existence and uniqueness of the characteristic curves defined by the ordinary differential

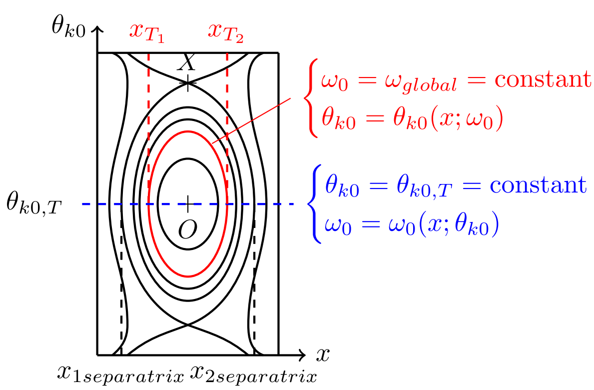

FIG. 4. Cat's eye picture.

equations (86). Let us now find the fixed point of the force field $F := (\partial_x \mathcal{H}_{\omega_0}, -\partial_{\theta_{k0}} \mathcal{H}_{\omega_0})^T$, i.e., the critical point of the Hamiltonian $\mathcal{H}_{\omega_0}$. Since the kernel of integral operator (71) is 2π-periodic in θ_{k0}, it is straightforward to prove that $\mathcal{H}_{\omega_0}$ is also 2π-periodic in θ_{k0} (see Remark 19). Therefore, there exists an infinite sequence of isolated points $\{\theta_{k0c,i}\}_{i\in\mathbb{N}}$ such that, for all $x \in [x_{\min}, x_{\max}]$,

$$(\partial_{\theta_{k0}} \mathcal{H}_{\omega_0})(\theta_{k0c,i}, x) = 0 \quad \text{and} \quad \text{sign}\left((\partial^2_{\theta_{k0}} \mathcal{H}_{\omega_0})(\theta_{k0c,i}, x)\right) = (-1)^i,\ i \in \mathbb{N}. \tag{87}$$

This sequence corresponds to the alternating sequence of maxima and minima of $\mathcal{H}_{\omega_0}$ in the variable θ_{k0}. Using the above and assumption *(i)* of Proposition 4, the only critical points are the sequence $\{(\theta_{k0c,i}, x_c)\}_{i\in\mathbb{N}}$. To determine the nature of these critical points, we need to study the eigenvalues of the gradient matrix ∇F at these critical points. It is straightforward to show that

$$\nabla F(\theta_{k0c,i}, x_c) = \begin{pmatrix} \partial^2_{\theta_{k0},x} \mathcal{H}_{\omega_0}, & -\partial^2_{\theta_{k0}} \mathcal{H}_{\omega_0} \\ \partial^2_x \mathcal{H}_{\omega_0}, & -\partial^2_{\theta_{k0},x} \mathcal{H}_{\omega_0} \end{pmatrix}$$

and

$$\det(\nabla F - \lambda I) = \lambda^2 - (\partial^2_{\theta_{k0},x} \mathcal{H}_{\omega_0})^2 + \partial^2_{\theta_{k0}} \mathcal{H}_{\omega_0} \partial^2_x \mathcal{H}_{\omega_0}. \tag{88}$$

Therefore, the eigenvalues λ of $\nabla F(\theta_{k0c,i}, x_c)$ satisfy $\lambda^2 = (\partial^2_{\theta_{k0},x} \mathcal{H}_{\omega_0})^2 - \partial^2_{\theta_{k0}} \mathcal{H}_{\omega_0} \partial^2_x \mathcal{H}_{\omega_0}$. On the one hand, if $(\partial^2_{\theta_{k0},x} \mathcal{H}_{\omega_0})^2 \geq \partial^2_{\theta_{k0}} \mathcal{H}_{\omega_0} \partial^2_x \mathcal{H}_{\omega_0}$, we have a saddle hyperbolic fixed point. If $\partial^2_x \mathcal{H}_{\omega_0}$ $(\theta_{k0c,i}, x_c) < 0$ (resp. $\partial^2_x \mathcal{H}_{\omega_0}(\theta_{k0c,i}, x_c) > 0$), then the latter inequality is satisfied at even (respectively, odd) points of the sequence $\{(\theta_{k0c,i}, x_c)\}_{i\in\mathbb{N}}$. On the other hand, if $(\partial^2_{\theta_{k0},x} \mathcal{H}_{\omega_0})^2 < \partial^2_{\theta_{k0}} \mathcal{H}_{\omega_0}$ $\partial^2_x \mathcal{H}_{\omega_0}$, we have a center, that is, an elliptic fixed point. If $\partial^2_x \mathcal{H}_{\omega_0}(\theta_{k0c,i}, x_c) < 0$ (resp. $\partial^2_x \mathcal{H}_{\omega_0}$ $(\theta_{k0c,i}, x_c) > 0$), then the latter inequality is satisfied at odd (resp. even) points of the sequence $\{(\theta_{k0c,i}, x_c)\}_{i\in\mathbb{N}}$, provided assumption *(ii)* holds. Therefore, we obtain an alternating sequence of X-points (saddle hyperbolic fixed points) and O-points (elliptic fixed points) along the line $x = x_c$. From this follows the rest of the stated results, which concludes the proof. □

Now, restricting the phase-space to a period in θ_{k0}, we obtain a periodic (θ_{k0}, x)-patch (see Fig. 4). The points $\mathbf{z} = (\theta_{k0}, x) = (\theta_{k0}, x(\theta_{k0}))$ on an orbit are two-valued functions of θ_{k0}, while conversely the points $\mathbf{z} = (\theta_{k0}, x) = (\theta_{k0}(x), x)$ of the same orbit are two-valued functions of x. Therefore, there are two points of view to describe the problem.

The first one is to view ω_0 as a global parameter and $\theta_{k0}(x;\omega_0)$ as a function of x, which depends parametrically on ω_0: for a given magnetic flux surface $x = x_*$, we search two branches $\theta^{\pm}_{k0,\omega_0}(x_*)$ such that $\mathcal{H}(\omega_0, \theta^+_{k0}(x_*;\omega_0), x_*) = \mathcal{H}(\omega_0, \theta^-_{k0}(x_*;\omega_0), x_*)$. The second one is to view θ_{k0} as a global parameter and $\omega_0(x;\theta_{k0})$ as a function of x, which depends parametrically on θ_{k0}: for a given θ_{k0} we search two magnetic flux surfaces x_1 and x_2 such that $\omega_0(x_1;\theta_{k0}) = \omega_0(x_2;\theta_{k0}) =$ constant. Let us consider the first point of view, which means that we search the characteristic curves of the phase-space constituted by the set of points $(\theta_{k0}(x;\omega_0), x)_{\omega_0}$ such that $\mathcal{H}(\omega_0, \theta_{k0}$ $(x;\omega_0), x) = 0$; each orbit being associated to a unique value of ω_0. Supposing that the trajectories are regular enough, the implicit function theorem implies that the function $\theta_{k0}(x;\omega_0)$ is implicitly defined by (85), as long as $\partial_{\theta_{k0}} \mathcal{H} \neq 0$. Taking the derivative of (85) with respect to x, we then find that $\partial_x \theta_{k0} = -\partial_x \mathcal{H} / \partial_{\theta_{k0}} \mathcal{H}$. Furthermore, validity of the eikonal representation (35) and (47) requires the condition $|\partial_x \theta_{k0}/(nq'\theta^2_{k0})| \ll 1$ to be satisfied. Nevertheless, from orbit topology in phase-space, for a periodic patch (see Fig. 4), we know that there exist two points x_{T_i}, $i \in \{1,2\}$, called turning points, such that $\partial_{\theta_{k0}} \mathcal{H}(\omega_0, \theta_{k0}(x_{T_i};\omega_0), x_{T_i}) = 0$ and $|\partial_x \theta_{k0}(x_{T_i};\omega_0)| = \infty$ (or $\partial_{\theta_{k0}} x(\theta_{k0,T}) = 0$), and where $\theta_{k0,T} = \theta_{k0}(x_{T_2};\omega_0) = \theta_{k0}(x_{T_1};\omega_0)$.

To avoid the turning points problem at zeroth order, we assume that $\partial_x \theta_{k0}(x) = 0$. Thus $\theta_{k0} =$ constant, and then we adopt the second point of view according to which θ_{k0} is a global parameter. Actually, the problem of dealing with the turning points is just postponed to the next order in the expansion of θ_k, i.e., for $\theta_{k1}(x)$. Therefore the dispersion equation $\mathcal{H}(\omega_0, \theta_{k0}, x) = 0$ can be interpreted as follows: we look for the local frequency $\omega_0(x;\theta_{k0})$ such that $\mathcal{H}(\omega_0(x;\theta_{k0}), \theta_{k0}, x) = 0$. As a consequence ω_0 is a function of x which depends parametrically on θ_{k0} and is implicitly defined by $\mathcal{H}(\omega_0(x;\theta_{k0}), \theta_{k0}, x) = 0$. The equation $\omega_0(x;\theta_{k0}) =$ constant should allow us to construct the phase-space orbits described by Hamilton's equations (86).

Within the second point of view let us show that the eigenvalue ω_0 reaches an extremum at the turning points $\theta_{k0,T}$ and that the first-order correction ω_1 to the eigenfrequency vanishes by the following proposition.

Proposition 5. Let us suppose that assumptions of Proposition 4 are satisfied. Let $\phi_{10\omega n}$ and $\widehat{\phi}_{10\omega n}$ be, respectively, the solution of the problem $\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n} = 0$ and the adjoint problem $\mathcal{L}^{\circ\star}_{\mathscr{C}\omega n}$ $\widehat{\phi}_{10\omega n} = 0$. We assume that

$$\left\langle \widehat{\phi}_{10\omega n}, \partial_{\omega_0}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n}\right\rangle_{L^2} \neq 0, \quad \forall\theta_{k0} \in \mathbb{R}/2\pi\mathbb{Z},\ \forall x \in [x_{\min}, x_{\max}]. \tag{89}$$

Then,

(i) *there exists a value of θ_{k0}, called a turning point and noted $\theta_{k0,T}$, such that*

$$\frac{\partial\omega_0}{\partial\theta_{k0}}(x;\theta_{k0,T}) = 0, \quad \forall x \in [x_{\min}, x_{\max}], \tag{90}$$

i.e., the eigenvalue ω_0 reaches an extremum at the turning points $\theta_{k0,T}$,

(ii) *the eigenvalue ω_0 reaches an extremum at $x = x_c$, i.e.,*

$$\frac{\partial\omega_0}{\partial x}(x_c;\theta_{k0}) = 0, \quad \forall\theta_{k0} \in \mathbb{R}/2\pi\mathbb{Z}, \tag{91}$$

(iii) *at the turning point $\theta_{k0} = \theta_{k0,T}$ we get, for the first-order system (76), the solution*

$$\omega_1 = 0, \tag{92}$$

$$\phi_{11\omega n} = \theta_{k1}\partial_{\theta_{k0}}\phi_{10\omega n} + g_1(x)\phi_{10\omega n}, \tag{93}$$

$$h^{\pm}_{1\mu b\omega n} = \theta_{k1}\partial_{\theta_{k0}}h^{\pm}_{0\mu b\omega n} + g_1(x)h^{\pm}_{0\mu b\omega n}, \tag{94}$$

with $g_1(x)$ an arbitrary function.

Remark 13. Even if condition (90) seems to restrict the search of the eigenvalues to a subset of the spectrum, it is actually not the case. Keeping in mind the topology of the characteristic curves in phase-space (the integrable cat's eye picture, see Fig. 4), the equation $\theta_{k0} = \theta_{k0,T}$ corresponds to the line passing through the O-point. Therefore we can deduce two important conclusions. Along this line (of equation $\theta_{k0} = \theta_{k0,T}$), if x is varying in its range, then ω_0 describes all the eigenvalues of the spectrum. Moreover the line $\theta_{k0} = \theta_{k0,T}$, corresponds to the value of θ_{k0} where the radial extension of ω_0 is maximum. As a consequence, it will select the eigenmodes for which the radial extension is maximum. Indeed, the radial extension of ω_0 will fix the radial extension of the potential function involved in the one-dimensional Schrödinger equation, which will determine the radial envelope of the eigenmode. Therefore, condition (90) allows us to recover all the eigenvalues of the spectrum, but only yields the eigenmodes of maximum radial extension.

Remark 14. The conditions (90) and (91) express that ω_0 reaches an extremal value (which should be a maximum in the case of an instability) at the point $(x_c, \theta_{k0,T})$. From θ-symmetry of the equilibrium and since Toroidal-ITG instability has maximum amplitude on the low-field side, we expect $\theta_{k0,T}$ to be close to the origin. Moreover we expect that x_c is located in the vicinity of the maximum of density and temperature gradients ($r = r_0$ by usual assumption), i.e., close to the origin (in the radial variable x).

Proof of Proposition 5. Let us start with first item *(i)* of Proposition 5. We first notice that at the turning point $\theta_{k0,T}$, we have the identity

$$\begin{aligned}\left\langle \widehat{\phi}_{10\omega n}, \partial_{\theta_{k0}}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n}\right\rangle_{L^2} &= \frac{\partial\mathscr{H}}{\partial\theta_{k0}} - \left\langle \partial_{\theta_{k0}}\widehat{\phi}_{10\omega n}, \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n}\right\rangle_{L^2} - \left\langle \widehat{\phi}_{10\omega n}, \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\partial_{\theta_{k0}}\phi_{10\omega n}\right\rangle_{L^2}\\ &= 0.\end{aligned}$$

Differentiating now (71) with respect to θ_{k0}, and taking the Hermitian product with $\widehat{\phi}_{10\omega n}$, and using the previous identity, we get

$$\frac{\partial\omega_0}{\partial\theta_{k0}}\left\langle \widehat{\phi}_{10\omega n}, \partial_{\omega_0}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n}\right\rangle_{L^2} = 0.$$

Using assumption (89) we finally obtain (90).

Let us continue with the second item *(ii)*. Using assumption *(i)* of Proposition 4, we have, at the point $x = x_c$,

$$\left\langle \widehat{\phi}_{10\omega n}, \partial_x \mathcal{L}^{\circ}_{\mathscr{C}\omega n} \phi_{10\omega n} \right\rangle_{L^2} = \frac{\partial \mathscr{H}}{\partial x} - \left\langle \partial_x \widehat{\phi}_{10\omega n}, \mathcal{L}^{\circ}_{\mathscr{C}\omega n} \phi_{10\omega n} \right\rangle_{L^2} - \left\langle \widehat{\phi}_{10\omega n}, \mathcal{L}^{\circ}_{\mathscr{C}\omega n} \partial_x \phi_{10\omega n} \right\rangle_{L^2} = 0. \quad (95)$$

Differentiating (71) with respect to x, and taking the Hermitian product with $\widehat{\phi}_{10\omega n}$, and using (95) we obtain, at the point $x = x_c$,

$$\frac{\partial \omega_0}{\partial x} \left\langle \widehat{\phi}_{10\omega n}, \partial_{\omega_0} \mathcal{L}^{\circ}_{\mathscr{C}\omega n} \phi_{10\omega n} \right\rangle_{L^2} = 0,$$

which shows item *(ii)*, under assumption (89).

It now remains to prove the third item *(iii)*. Until now, we have described the fast and slow variations in the (θ,φ)-variables and fast variation in the r-variable. We now want to solve for the intermediate scale of variation of length order $n^{-\sigma} a$ in the radial direction, which corresponds to the radial extension of the eigenmode. Since we are interested in the eigenmodes of maximum radial extension, we fix $\theta_{k0} = \theta_{k0,T}$. Therefore, we restrict our problem to finding the spectrum of the integral operator $\mathcal{L}^{\circ}_{\mathscr{C}\omega n}$ for which (90) is satisfied. Taking the Hermitian product of (76) with $\widehat{\phi}_{10\omega n}$, and using

$$\left\langle \widehat{\phi}_{10\omega n}, \partial_{\theta_{k0}} \mathcal{L}^{\circ}_{\mathscr{C}\omega n} \phi_{10\omega n} \right\rangle_{L^2} + \frac{\partial \omega_0}{\partial \theta_{k0}} \left\langle \widehat{\phi}_{10\omega n}, \partial_{\omega_0} \mathcal{L}^{\circ}_{\mathscr{C}\omega n} \phi_{10\omega n} \right\rangle_{L^2} = 0,$$

we obtain

$$\left(\omega_1 - \theta_{k1} \frac{\partial \omega_0}{\partial \theta_{k0}} \right) \left\langle \widehat{\phi}_{10\omega n}, \partial_{\omega_0} \mathcal{L}^{\circ}_{\mathscr{C}\omega n} \phi_{10\omega n} \right\rangle_{L^2} - \frac{\mathrm{i}}{2} \frac{1}{nq'} \partial_x \theta_{k0} \partial^2_{\theta_{k0}} Q_{0\omega n} \left\langle \widehat{\phi}_{10\omega n}, \phi_{10\omega n} \right\rangle_{L^2} = 0.$$

Using (89) and (90), the previous equation leads to (92).

Let us now establish formula (93). Differentiating (71) with respect to θ_{k0} and using (90), we get

$$\partial_{\theta_{k0}} \mathcal{L}^{\circ}_{\mathscr{C}\omega n} \phi_{10\omega n} = -\mathcal{L}^{\circ}_{\mathscr{C}\omega n} \partial_{\theta_{k0}} \phi_{10\omega n}. \quad (96)$$

From (76), since $\omega_1 = 0$ and $\partial_x \theta_{k0} = 0$, we get

$$\mathcal{L}^{\circ}_{\mathscr{C}\omega n} \phi_{11\omega n} + \theta_{k1} \partial_{\theta_{k0}} \mathcal{L}^{\circ}_{\mathscr{C}\omega n} \phi_{10\omega n} = 0.$$

Using (96), we can solve the previous equation to obtain (93) with $g_1(x)$ an arbitrary function. It now remains to establish (94). Let us first do this for open contours and next for closed contours. Using (82) and (83) and (92) and (93), and taking the θ_{k0}-derivative of (72), we obtain

$$\begin{aligned} \theta_{k1} \partial_{\theta_{k0}} h^{\pm}_{0\mu b\omega n} = &\mp \mathrm{i} \int_{\mp\infty}^{\eta} d\widehat{\eta} \left(\frac{qR}{b_\varphi} \left[\beta_{\mu b} - \frac{q_i}{m_i} \frac{\alpha_{\mu b}}{a_{\mu b}} \right] \phi_{10\omega n} + \gamma_{\mu b} \phi_{11\omega n} \right)(\widehat{\eta})\, \mathcal{K}_{\mu b}(\eta, \widehat{\eta}) \\ &\mp \mathrm{i} \int_{\mp\infty}^{\eta} d\widehat{\eta}\, (\gamma_{\mu b} \phi_{10\omega n})(\widehat{\eta}) \mathcal{K}_{\mu b}(\eta, \widehat{\eta})\, [-\mathrm{i}\, \mathrm{sign}(\eta - \widehat{\eta})] \int_{\widehat{\eta}}^{\eta} d\widetilde{\eta} \left(\frac{qR}{b_\varphi} \frac{\alpha_{\mu b}}{a_{\mu b}} \right)(\widetilde{\eta}). \end{aligned}$$

Using Fubini's theorem to permutate the order of integrations, in the second term on the right-hand side of the previous equation, we get the relation (94) for open contours. Let us now deal with closed contours by following the same method that we used for open contours. Differentiating (80) with respect to θ_{k0}, using (82) and (83) and (92) and (93) and Fubini's theorem to permutate the order of η-integrations in some double integrals, we obtain

$$\begin{aligned} \theta_{k1} \partial_{\theta_{k0}} h^{\pm}_{0\mu b\omega n} &= \theta_{k1} \partial_{\theta_{k0}} h^{\pm}_{0\mu b\omega n}(\theta^1_{L\mu b}) \exp\left(\pm \mathrm{i} \mathcal{I}_{\mu b}(\theta^1_{L\mu b}, \eta) \right) \\ &\mp \mathrm{i} \int_{\theta^1_{L\mu b}}^{\eta} d\widehat{\eta} \left(\frac{qR}{b_\varphi} \beta_{\mu b} \phi_{10\omega n} + \gamma_{\mu b} \phi_{11\omega n} + \frac{qR}{b_\varphi} \frac{\alpha_{\mu b}}{a_{\mu b}} \left[h^{\pm}_{0\mu b\omega n} - \frac{q_i}{m_i} \phi_{10\omega n} \right] \right)(\widehat{\eta}) \exp\left(\mp \mathrm{i} \mathcal{I}_{\mu b}(\eta, \widehat{\eta}) \right). \end{aligned} \quad (97)$$

Now, the θ_{k0}-derivative of the right-hand side of (74) gives

$$\begin{aligned}
&\theta_{k1}\partial_{\theta_{k0}} h^{\pm}_{0\mu b\omega n}(\theta^1_{L\mu b})\\
&= \theta_{k1}\Bigg\{-\frac{\cos\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\theta^2_{L\mu b})}{\sin\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\theta^2_{L\mu b})}\partial_{\theta_{k0}}\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\theta^2_{L\mu b})\int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\widetilde{\eta}\,(\gamma_{\mu b}\phi_{10\omega n})(\widetilde{\eta})\cos\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\widetilde{\eta})\\
&\quad + \sin^{-1}\mathcal{I}_{\mu b}(\theta^1_{L\mu b},\theta^2_{L\mu b})\int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\widetilde{\eta}\,\Bigg\{\left(\gamma_{\mu b}\partial_{\theta_{k0}}\phi_{10\omega n} + \frac{qR}{b_\varphi\theta_{k1}}\left[\beta_{\mu b} - \frac{q_i}{m_i}\frac{\alpha_{\mu b}}{a_{\mu b}}\right]\phi_{10\omega n}\right)(\widetilde{\eta})\\
&\qquad \cos\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\widetilde{\eta}) + \left(\gamma_{\mu b}\phi_{10\omega n}\right)(\widetilde{\eta})\partial_{\theta_{k0}}\left[\cos\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\widetilde{\eta})\right]\Bigg\}\Bigg\}. \qquad (98)
\end{aligned}$$

Using Fubini's theorem to permutate the order of η-integrations in some double integrals, we get that

$$\begin{aligned}
&\int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\widetilde{\eta}\,\left(\gamma_{\mu b}\phi_{10\omega n}\right)(\widetilde{\eta})\,\partial_{\theta_{k0}}\left[\cos\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\widetilde{\eta})\right]\\
&\qquad = -\int_{\theta^1_{L\mu b}}^{\theta^2_{L\mu b}} d\widehat{\eta}\,\partial_{\theta_{k0}}\partial_\eta\mathcal{I}_{\mu b}(\widehat{\eta},\eta_0)\int_{\theta^1_{L\mu b}}^{\widehat{\eta}} d\widetilde{\eta}\left(\gamma_{\mu b}\phi_{10\omega n}\right)(\widetilde{\eta})\sin\mathcal{I}_{\mu b}(\theta^2_{L\mu b},\widetilde{\eta}).
\end{aligned}$$

Inserting the last expression into (98), we get $\theta_{k1}\partial_{\theta_{k0}}h^{\pm}_{0\mu b\omega n}(\theta^1_{L\mu b}) = h^{\pm}_{1\mu b\omega n}(\theta^1_{L\mu b})$ which together with (97) implies (94) for closed contours. □

We are now ready to solve the second-order problem to determine θ_{k1} and thus obtain the radial envelope of the eigenmode.

D. The second-order system

In this section we solve the system (69) and (70), which leads to the determination of the complex function $[q_{\min}, q_{\max}] \ni q \longmapsto \theta_{k1}(q) \in \mathbb{C}$ satisfying a Riccati equation or equivalently, of the complex amplitude

$$A_1(q) = \exp\left(\mathrm{i}n\int_{q_0}^{q} d\tilde{q}\,\theta_{k1}(\tilde{q})\right)A_1(q_0) = \exp\left(\mathrm{i}n\int_{q_0}^{q} d\tilde{q}\,\Theta_{k1}\right)A_1(q_0),$$

satisfying a Schrödinger equation. With the definition $\Theta_{k1} = -(\mathrm{i}/n)\partial_q$ (see Remark 9 of Sec. III C 2), we have the following.

Proposition 6. Let $1/\gamma > 2$. Then the system (69) and (70) is equivalent to solving the non-self-adjoint Schrödinger equation

$$\left(-\Theta^2_{k1} + \frac{\omega - \omega_0}{\frac{1}{2}\partial^2_{\theta_{k0}}\omega_0}\right)A_1 = 0, \qquad (99)$$

for the complex amplitude A_1 or equivalently the Riccati equation

$$\frac{\mathrm{i}}{n}\partial_q\theta_{k1} - \theta^2_{k1} + \frac{\omega - \omega_0}{\frac{1}{2}\partial^2_{\theta_{k0}}\omega_0} = 0, \qquad (100)$$

for the complex function θ_{k1}.

Proof. Here, we restrict our problem to the determination of the radial envelope of the eigenmode whose scale of variation is of length order $n^{-\sigma}a$, with $\sigma = 1 - \gamma$ and $1/\gamma > 2$. The calculation is tedious, so for simplicity, we only give the full derivation for the set of open contours. For the set of closed contours calculations are similar, and lead to the same result as for open contours. Before doing so, let us introduce the following notation:

$$\xi_{\mu b} = -\mathrm{i}\left(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0}\right)\mathcal{L}_{0\mu b\omega n},$$

$$\zeta_{\mu b} = -\mathrm{i}\left(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0}\right)\mathcal{M}_{0\mu b\omega n},$$
$$\gamma_{\mu b} = \frac{e}{T_{e0}}\frac{qR}{b_\varphi}\Omega_{*\mu b}a_{\mu b},$$
$$\mathcal{K}_{\mu b}(\eta,\widehat{\eta}) = \exp\left(-\mathrm{i}\,\mathrm{sign}(\eta-\widehat{\eta})\mathcal{I}_{\mu b}(\eta,\widehat{\eta})\right).$$

Following the same method that we used for dealing with the zeroth- and first-order systems and integrating (69) with respect to the η-variable, we obtain, for open contours,

$$h^{\pm}_{2\mu b\omega n} = -\mathrm{i}\int_{\mp\infty}^{\eta} d\widehat{\eta}\,\Bigg(\pm\gamma_{\mu b}\phi_{12\omega n} \pm \frac{qR}{b_\varphi}\Bigg[\left\{\frac{\alpha_{\mu b}}{a_{\mu b}}(\theta_{k1}\partial_{\theta_{k0}} + g_1) + \frac{\xi_{\mu b}}{a_{\mu b}}\right\}h^{\pm}_{0\mu b\omega n}$$
$$+\left\{\left(\beta_{\mu b} - \frac{q_i}{m_i}\frac{\alpha_{\mu b}}{a_{\mu b}}\right)(\theta_{k1}\partial_{\theta_{k0}} + g_1) + \zeta_{\mu b} - \frac{q_i}{m_i}\frac{\xi_{\mu b}}{a_{\mu b}}\right\}\phi_{10\omega n}\Bigg]\Bigg)(\widehat{\eta})\,\mathcal{K}_{\mu b}(\eta,\widehat{\eta}), \quad (101)$$

where we have used (93) and (94). Using (101), we get

$$h^{+}_{2\mu b\omega n} + h^{-}_{2\mu b\omega n} = -\mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\Bigg(\gamma_{\mu b}\phi_{12\omega n} + \Bigg\{\left(\beta_{\mu b} - \frac{q_i}{m_i}\frac{\alpha_{\mu b}}{a_{\mu b}}\right)(\theta_{k1}\partial_{\theta_{k0}} + g_1)$$
$$+\zeta_{\mu b} - \frac{q_i}{m_i}\frac{\xi_{\mu b}}{a_{\mu b}}\Bigg\}\phi_{10\omega n}\Bigg)(\widehat{\eta})\,\mathcal{K}_{\mu b}(\eta,\widehat{\eta}) - \mathrm{i}(X^{+}_{\mu b} + X^{-}_{\mu b}), \quad (102)$$

where

$$X^{\pm}_{\mu b} = \pm\int_{\mp\infty}^{\eta} d\widehat{\eta}\,\left(\frac{qR}{b_\varphi}\frac{1}{a_{\mu b}}\left[\alpha_{\mu b}(\theta_{k1}\partial_{\theta_{k0}} + g_1) + \xi_{\mu b}\right]h^{\pm}_{0\mu b\omega n}\right)(\widehat{\eta})\,\mathcal{K}_{\mu b}(\eta,\widehat{\eta})$$
$$= \mp\mathrm{i}\int_{\mp\infty}^{\eta} d\widehat{\eta}\int_{\mp\infty}^{\widehat{\eta}} d\widetilde{\eta}\,\Bigg[\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\mathcal{K}_{\mu b}(\widehat{\eta},\widetilde{\eta})\left(\frac{qR}{b_\varphi}\frac{1}{a_{\mu b}}(\alpha_{\mu b}g_1 + \xi_{\mu b})\right)(\widehat{\eta})(\gamma_{\mu b}\phi_{10\omega n})(\widetilde{\eta}) + \mathcal{K}_{\mu b}(\eta,\widetilde{\eta})$$
$$\left(\frac{qR}{b_\varphi}\frac{\alpha_{\mu b}}{a_{\mu b}}\right)(\widehat{\eta})\,\theta_{k1}\left\{\partial_{\theta_{k0}}(\gamma_{\mu b}\phi_{10\omega n})(\widetilde{\eta}) - \mathrm{i}\,\mathrm{sign}(\widehat{\eta}-\widetilde{\eta})(\gamma_{\mu b}\phi_{10\omega n})(\widetilde{\eta})\partial_{\theta_{k0}}\mathcal{I}_{\mu b}(\widehat{\eta},\widetilde{\eta})\right\}\Bigg]. \quad (103)$$

Using now Fubini's theorem to permutate the order of η-integrations in the double integral, (103) becomes

$$X^{\pm}_{\mu b} = \mp\mathrm{i}\int_{\mp\infty}^{\eta} d\widetilde{\eta}\,\Bigg\{\left(g_1\gamma_{\mu b}\phi_{10\omega n} + \theta_{k1}\partial_{\theta_{k0}}[\gamma_{\mu b}\phi_{10\omega n}]\right)(\widetilde{\eta})\,\mathrm{sign}(\eta-\widetilde{\eta})\mathcal{K}_{\mu b}(\eta,\widetilde{\eta})\int_{\widetilde{\eta}}^{\eta} d\widehat{\eta}\left(\frac{qR}{b_\varphi}\frac{\alpha_{\mu b}}{a_{\mu b}}\right)(\widehat{\eta})$$
$$+(\gamma_{\mu b}\phi_{10\omega n})(\widetilde{\eta})\,\mathrm{sign}(\eta-\widetilde{\eta})\mathcal{K}_{\mu b}(\eta,\widetilde{\eta})\int_{\widetilde{\eta}}^{\eta} d\widehat{\eta}\left(\frac{qR}{b_\varphi}\frac{\xi_{\mu b}}{a_{\mu b}}\right)(\widehat{\eta})\Bigg\} + Y^{\pm}_{\mu b}, \quad (104)$$

where, by (82), we have

$$Y^{\pm}_{\mu b} = \mp\int_{\mp\infty}^{\eta} d\widetilde{\eta}\,\theta_{k1}(\gamma_{\mu b}\phi_{10\omega n})(\widetilde{\eta})\,\mathrm{sign}(\eta-\widetilde{\eta})\mathcal{K}_{\mu b}(\eta,\widetilde{\eta})\int_{\widetilde{\eta}}^{\eta} d\widehat{\eta}\,\partial_{\theta_{k0}}\mathcal{I}_{\mu b}(\widehat{\eta},\widetilde{\eta})\left(\frac{qR}{b_\varphi}\frac{\alpha_{\mu b}}{a_{\mu b}}\right)(\widehat{\eta})$$
$$= \mp\int_{\mp\infty}^{\eta} d\widetilde{\eta}\,\theta_{k1}^2(\gamma_{\mu b}\phi_{10\omega n})(\widetilde{\eta})\mathcal{K}_{\mu b}(\eta,\widetilde{\eta})\int_{\widetilde{\eta}}^{\eta} d\widehat{\eta}\,\partial_{\theta_{k0}}\mathcal{I}_{\mu b}(\widehat{\eta},\widetilde{\eta})\partial_{\widehat{\eta}}\partial_{\theta_{k0}}\mathcal{I}_{\mu b}(\widehat{\eta},\widetilde{\eta})$$
$$= \mp\int_{\mp\infty}^{\eta} d\widehat{\eta}\,\frac{\theta_{k1}^2}{2}(\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\left[\partial_{\theta_{k0}}\mathcal{I}_{\mu b}(\eta,\widehat{\eta})\right]^2.$$

Substituting the previous equation into (104) and thus into (103), using (82) and (92), and in addition noting that

$$\xi_{\mu b} = \frac{b_\varphi a_{\mu b}}{qR}\left(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0}\right)\partial_\eta\mathcal{I}_{\mu b}(\eta,\eta_0), \quad \forall\eta_0\in\mathbb{R},$$
$$\zeta_{\mu b} = \frac{b_\varphi}{qR}\left(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0}\right)\gamma_{\mu b} + \frac{q_i}{m_i}\frac{\xi_{\mu b}}{a_{\mu b}},$$

we find that (102) becomes:

$$h^+_{2\mu b\omega n} + h^-_{2\mu b\omega n} = -\mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\left(\gamma_{\mu b}\phi_{12\omega n} + \{\theta_{k1}\partial_{\theta_{k0}}\gamma_{\mu b}\,[\theta_{k1}\partial_{\theta_{k0}} + g_1]\right.$$
$$\left. + \left[\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0}\right]\gamma_{\mu b}\}\,\phi_{10\omega n}\right)(\widehat{\eta})\,\mathcal{K}_{\mu b}(\eta,\widehat{\eta}) - \mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\left[g_1(\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\right.$$
$$\left. + \theta_{k1}\partial_{\theta_{k0}}(\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\right]\{-\mathrm{i}\,\mathrm{sign}(\eta-\widehat{\eta})\}\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\int_{\widehat{\eta}}^{\eta} d\widetilde{\eta}\,\theta_{k1}\partial_{\theta_{k0}}\partial_{\widetilde{\eta}}\mathcal{I}_{\mu b}(\widetilde{\eta},\eta_0)$$
$$-\,\mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,(\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\{-\mathrm{i}\,\mathrm{sign}(\eta-\widehat{\eta})\}\int_{\widehat{\eta}}^{\eta} d\widetilde{\eta}\,\left(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0}\right)\partial_{\widetilde{\eta}}\mathcal{I}_{\mu b}(\widetilde{\eta},\eta_0)$$
$$+\,\mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\frac{\theta_{k1}^2}{2}(\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\left[\partial_{\theta_{k0}}\mathcal{I}_{\mu b}(\eta,\widehat{\eta})\right]^2,$$

which after integration leads to

$$h^+_{2\mu b\omega n} + h^-_{2\mu b\omega n} = -\mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\Big[\left(\gamma_{\mu b}\phi_{12\omega n}\right)(\widehat{\eta})\,\mathcal{K}_{\mu b}(\eta,\widehat{\eta})$$
$$+\,\phi_{10\omega n}(\widehat{\eta})\left(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0}\right)\{\gamma_{\mu b}(\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\} + \phi_{11\omega n}(\widehat{\eta})\theta_{k1}\partial_{\theta_{k0}}\{\gamma_{\mu b}(\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\}\Big]$$
$$-\,\mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\theta_{k1}^2\left[(\partial_{\theta_{k0}}\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\partial_{\theta_{k0}}\mathcal{K}_{\mu b}(\eta,\widehat{\eta}) - \frac{1}{2}(\gamma_{\mu b}\phi_{10\omega n})(\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\{\partial_{\theta_{k0}}\mathcal{I}_{\mu b}(\eta,\widehat{\eta})\}^2\right]. \tag{105}$$

Using now the fact that

$$\partial^2_{\theta_{k0}}\gamma_{\mu b} = 0, \quad \text{and} \quad \partial^2_{\theta_{k0}}\mathcal{K}_{\mu b}(\eta,\widehat{\eta}) = -\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\{\partial_{\theta_{k0}}\mathcal{I}_{\mu b}(\eta,\widehat{\eta})\}^2 \quad (\text{since } \partial^2_{\theta_{k0}}\mathcal{I}_{\mu b}(\eta,\widehat{\eta}) = 0),$$

Equation (105) can be recast as

$$h^+_{2\mu b\omega n} + h^-_{2\mu b\omega n} =$$
$$-\,\mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\left(\gamma_{\mu b}\phi_{12\omega n}\right)(\widehat{\eta})\,\mathcal{K}_{\mu b}(\eta,\widehat{\eta}) - \mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\phi_{10\omega n}(\widehat{\eta})\left(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0}\right)\{\gamma_{\mu b}(\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\}$$
$$-\,\mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\phi_{11\omega n}(\widehat{\eta})\theta_{k1}\partial_{\theta_{k0}}\{\gamma_{\mu b}(\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\} - \mathrm{i}\int_{-\infty}^{+\infty} d\widehat{\eta}\,\frac{\theta_{k1}^2}{2}\phi_{10\omega n}(\widehat{\eta})\partial^2_{\theta_{k0}}\{\gamma_{\mu b}(\widehat{\eta})\mathcal{K}_{\mu b}(\eta,\widehat{\eta})\},$$

from which we obtain, for open contours,

$$-2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in O}\frac{\mathcal{A}_{\mu b}}{a_{\mu b}}(h^+_{2\mu b\omega n} + h^-_{2\mu b\omega n}) = -\mathcal{L}^{\circ}_{O\omega n}\phi_{12\omega n}$$
$$-\,(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0})\mathcal{L}^{\circ}_{O\omega n}\phi_{10\omega n} - \theta_{k1}\partial_{\theta_{k0}}\mathcal{L}^{\circ}_{O\omega n}\phi_{11\omega n} - \frac{\theta_{k1}^2}{2}\partial^2_{\theta_{k0}}\mathcal{L}^{\circ}_{O\omega n}\phi_{10\omega n}.$$

Following the same calculations developed for open contours, we obtain for closed contours

$$-\,2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in C}\frac{\mathcal{A}_{\mu b}}{a_{\mu b}}(h^+_{2\mu b\omega n} + h^-_{2\mu b\omega n}) = -\mathcal{L}^{\circ}_{C\omega n}\phi_{12\omega n}$$
$$-\,(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0})\mathcal{L}^{\circ}_{C\omega n}\phi_{10\omega n} - \theta_{k1}\partial_{\theta_{k0}}\mathcal{L}^{\circ}_{C\omega n}\phi_{11\omega n} - \frac{\theta_{k1}^2}{2}\partial^2_{\theta_{k0}}\mathcal{L}^{\circ}_{C\omega n}\phi_{10\omega n}.$$

Substituting the last two equations into (70), we obtain

$$\mathcal{Q}^{\circ}_{\omega n}\phi_{12\omega n} + \mathcal{Q}_{11\omega n}\phi_{11\omega n} + \mathcal{Q}_{12\omega n}\phi_{10\omega n} + \mathcal{L}^{\circ}_{O\cup C\omega n}\phi_{12\omega n}$$
$$+\,(\theta_{k2}\partial_{\theta_{k0}} + \omega_2\partial_{\omega_0})\mathcal{L}^{\circ}_{O\cup C\omega n}\phi_{10\omega n} + \theta_{k1}\partial_{\theta_{k0}}\mathcal{L}^{\circ}_{O\cup C\omega n}\phi_{11\omega n} + \frac{1}{2}\theta_{k1}^2\partial^2_{\theta_{k0}}\mathcal{L}^{\circ}_{O\cup C\omega n}\phi_{10\omega n} = 0. \tag{106}$$

Observing that

$$\Theta_{k1}^2 A_1 = A_1\left(\theta_{k1}^2 - \frac{i}{n}\partial_q \theta_{k1}\right),$$

(see Remark 9 of Sec. III C 2) and using (63) and (64), we find that (106) is equivalent to

$$\Big(\partial_{\theta_{k0}}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\left\{\phi_{10\omega n}\Theta_{k2} + \partial_{\theta_{k0}}\phi_{10\omega n}\Theta_{k1}^2 + g_1\phi_{10\omega n}\Theta_{k1}\right\} + \frac{1}{2}\partial^2_{\theta_{k0}}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n}\Theta_{k1}^2 + \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{12\omega n} + \omega_2\partial_{\omega_0}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n}\Big)A_1 = 0. \quad (107)$$

Differentiating twice (71) with respect to θ_{k0} and using (90), we get

$$\partial_{\theta_{k0}}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\partial_{\theta_{k0}}\phi_{10\omega n} + \frac{1}{2}\partial^2_{\theta_{k0}}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n} = -\frac{1}{2}\left(\frac{\partial^2\omega_0}{\partial\theta_{k0}^2}\partial_{\omega_0}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n} + \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\partial^2_{\theta_{k0}}\phi_{10\omega n}\right).$$

Substituting the previous equation and (96) into (107), we obtain

$$\Bigg(\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{12\omega n} + \omega_2\partial_{\omega_0}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n} - \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\partial_{\theta_{k0}}\phi_{10\omega n}g_1\Theta_{k1} - \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\partial_{\theta_{k0}}\phi_{10\omega n}\Theta_{k2} - \frac{1}{2}\left\{\frac{\partial^2\omega_0}{\partial\theta_{k0}^2}\partial_{\omega_0}\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\phi_{10\omega n} + \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\partial^2_{\theta_{k0}}\phi_{10\omega n}\right\}\Theta_{k1}^2\Bigg)A_1 = 0.$$

Taking the Hermitian product of the last expression with $\widehat{\phi}_{10\omega n}$, using (52) and (92) to see that $\omega_2 = \omega - \omega_0 + O(\epsilon^{3\gamma})$, and taking into account the condition (89), we finally obtain the Schrödinger equation (99) or the Riccati equation (100). □

So far we have assumed for simplicity the gyroaverage operator to be the identity. We now turn to the more general case of a non-trivial gyroaverage operator.

Before summarizing the three stages of the asymptotic analysis done above to design an algorithm for computing the eigenmodes, we present in Sec. IV E how to extend the previous asymptotic analysis to the case including the gyroaverage operator.

E. Asymptotic analysis including a non-trivial gyroaverage operator

In this section we extend the previous asymptotic analysis to the case where we keep the gyroaverage operator $\mathfrak{J}_\mu$ (see (42)) in Equations (65)-(70).

Proposition 7. Let $1/\gamma > 2$. Then propositions 2–5 and 6 remain valid if we replace the linear operator $\mathcal{L}^{\circ}_{\mathscr{C}\omega n}$ by the gyroaveraged linear operator $\langle\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\rangle$, which is defined by

$$\langle\mathcal{L}^{\circ}_{\mathscr{C}\omega n}\rangle = \langle Q^{\circ}_{\omega n}\rangle + \langle\mathcal{L}^{\circ}_{O\omega n}\rangle + \langle\mathcal{L}^{\circ}_{C\omega n}\rangle,$$

where

$$\langle Q^{\circ}_{\omega n}\rangle = Q_{0\omega n} + 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\mathfrak{J}_{0\mu}\left(\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right)\mathfrak{J}_{0\mu},$$

$$\langle\mathcal{L}^{\circ}_{O\omega n}\rangle = \sum_{\mu b\in O}\mathfrak{J}_{0\mu}\mathcal{L}^{\circ}_{O\mu b\omega n}\mathfrak{J}_{0\mu}, \quad \langle\mathcal{L}^{\circ}_{C\omega n}\rangle = \sum_{\mu b\in C}\mathfrak{J}_{0\mu}\mathcal{L}^{\circ}_{C\mu b\omega n}\mathfrak{J}_{0\mu},$$

$$\mathfrak{J}_{0\mu} = J_0\left(\frac{|n|q}{r}\frac{v_\perp}{\Omega_i}\sqrt{1 + s^2(\eta - \theta_{k0})^2}\right).$$

Here, $s = q'r/q$ denotes the shear parameter, and J_0 the Bessel function of first kind of order zero.

Proof. The expansion of the gyroaverage operator $\mathfrak{J}_\mu$ (given by (42)) in powers of ϵ^γ is

$$\begin{aligned}\mathfrak{J}_\mu &= \mathfrak{J}_\mu(\theta_{k0}) + (\theta_k - \theta_{k0})\partial_{\theta_{k0}}\mathfrak{J}_\mu(\theta_{k0}) + \frac{1}{2}(\theta_k - \theta_{k0})^2\partial^2_{\theta_{k0}}\mathfrak{J}_\mu(\theta_{k0}) + \ \ldots \\ &= \mathfrak{J}_\mu(\theta_{k0}) + \theta_{k1}\partial_{\theta_{k0}}\mathfrak{J}_\mu(\theta_{k0}) + \left(\theta_{k2}\partial_{\theta_{k0}}\mathfrak{J}_\mu(\theta_{k0}) + \frac{1}{2}\theta_{k1}^2\partial^2_{\theta_{k0}}\mathfrak{J}_\mu(\theta_{k0})\right) + O(\epsilon^{3\gamma}). \quad (108)\end{aligned}$$

We also have to perform the asymptotic expansion of $\mathfrak{I}_\mu(\theta_{k0})$. Using the scale ordering of Sec. II D and the gyroaverage operator (42), we obtain

$$\mathfrak{I}_\mu(\theta_{k0})\psi(\eta,x) = \frac{1}{2\pi}\int_0^{2\pi} d\zeta\ \exp(\mathrm{i}\alpha(\zeta))\psi\left(\eta - (v_\perp/[\Omega_i r])\cos\zeta, x - (v_\perp/\Omega_i)\sin\zeta\right), \quad (109)$$

where we can make the decomposition $\alpha(\zeta) = \alpha_0(\zeta) + R(\zeta)$, with $\alpha_0(\zeta) = O(1)$, $R(\zeta) = O(\epsilon)$,

$$\alpha_0(\zeta) = \frac{nq}{r}\frac{v_\perp}{\Omega_i}\left(\cos\zeta + s(\eta - \theta_{k0})\sin\zeta\right),$$

and $s = q'r/q$. Therefore, from (109) we obtain

$$\mathfrak{I}_\mu(\theta_{k0})\psi(\eta,x) = \frac{1}{2\pi}\int_0^{2\pi} d\zeta\ \exp(\mathrm{i}\alpha_0(\zeta))\psi(\eta,x) + O(\epsilon).$$

Defining $\tan(\beta) = s(\eta - \theta_{k0})$, the last equation becomes

$$\begin{aligned}
\mathfrak{I}_\mu(\theta_{k0}) &= \frac{1}{2\pi}\int_0^{2\pi} d\zeta\ \exp(\mathrm{i}\alpha_0(\zeta)) + O(\epsilon)\\
&= \frac{1}{2\pi}\int_0^{2\pi} d\zeta\ \exp\left(\mathrm{i}\frac{nq}{r}\frac{v_\perp}{\Omega_i}\sqrt{1 + s^2(\eta-\theta_{k0})^2}\cos(\zeta-\beta)\right) + O(\epsilon)\\
&= J_0\left(\frac{|n|q}{r}\frac{v_\perp}{\Omega_i}\sqrt{1 + s^2(\eta-\theta_{k0})^2}\right) + O(\epsilon)\\
&= \mathfrak{I}_{0\mu} + O(\epsilon),
\end{aligned}$$

where J_0 is the Bessel function of first kind of order zero. Let us remember that to be consistent with the previous asymptotic analysis we can drop all the terms smaller than order $\epsilon^{2\gamma}$ with $\gamma < 1/2$. Therefore the asymptotic expansion (108) is still valid if we replace $\mathfrak{I}_\mu(\theta_{k0})$ by $\mathfrak{I}_{0\mu}$ in (108). Let us now introduce the following notation:

$$\begin{aligned}
\Phi_{10\omega n} &= \mathfrak{I}_{0\mu}\phi_{10\omega n},\\
\Phi_{11\omega n} &= \mathfrak{I}_{0\mu}\phi_{11\omega n} + \theta_{k1}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\phi_{10\omega n},\\
\Phi_{12\omega n} &= \mathfrak{I}_{0\mu}\phi_{12\omega n} + \theta_{k1}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\phi_{11\omega n} + \left(\theta_{k2}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu} + \frac{1}{2}\theta_{k1}^2\partial^2_{\theta_{k0}}\mathfrak{I}_{0\mu}\right)\phi_{10\omega n},
\end{aligned}$$

and for $j \in \{0,1,2\}$, $h^\pm_{j\mu b\omega n}[\Phi_{1j\omega n}] = w^\pm_{j\mu b\omega n}[\Phi_{1j\omega n}] + (q_i/m_i)\Phi_{1j\omega n}$. The bracket notation is used to emphasize that for $j \in \{0,1,2\}$, the perturbed Hamiltonians $h^\pm_{j\mu b\omega n}[\Phi_{1j\omega n}]$ are functionals of $\Phi_{1j\omega n}$ which are obtained by solving respectively (65), (67), and (69). Substituting the asymptotic expansions (49)-(51), (53)-(55), and (108) into (45) and (46), we get, at the zeroth order, the system

$$\mathcal{L}^\pm_{0\mu b\omega n}\left(\frac{h^\pm_{0\mu b\omega n}}{a^\pm_{\mu b}}\right) + \mathcal{M}^\pm_{0\mu b\omega n}\Phi_{10\omega n} - \frac{q_i}{m_i}\mathcal{L}^\pm_{0\mu b\omega n}\left(\frac{\Phi_{10\omega n}}{a^\pm_{\mu b}}\right) = 0, \quad (110)$$

$$\begin{aligned}
&\left(\mathcal{Q}_{0\omega n} + 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\mathfrak{I}_{0\mu}\right)\phi_{10\omega n}\\
&\quad = 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\mathfrak{I}_{0\mu}\left(\frac{h^+_{0\mu b\omega n}}{a^+_{\mu b}} - \frac{h^-_{0\mu b\omega n}}{a^-_{\mu b}}\right), \quad (111)
\end{aligned}$$

at the first order,

$$\begin{aligned}
&\mathcal{L}^\pm_{0\mu b\omega n}\left(\frac{h^\pm_{1\mu b\omega n}}{a^\pm_{\mu b}}\right) + \mathcal{M}^\pm_{0\mu b\omega n}\Phi_{11\omega n} - \frac{q_i}{m_i}\mathcal{L}^\pm_{0\mu b\omega n}\left(\frac{\Phi_{11\omega n}}{a^\pm_{\mu b}}\right)\\
&\quad + \mathcal{L}^\pm_{1\mu b\omega n}\left(\frac{h^\pm_{0\mu b\omega n}}{a^\pm_{\mu b}}\right) + \mathcal{M}^\pm_{1\mu b\omega n}\Phi_{10\omega n} - \frac{q_i}{m_i}\mathcal{L}^\pm_{1\mu b\omega n}\left(\frac{\Phi_{10\omega n}}{a^\pm_{\mu b}}\right) = 0, \quad (112)
\end{aligned}$$

$$\left(Q_{0\omega n} + 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\mathfrak{I}_{0\mu}\right)\phi_{11\omega n}$$
$$+\left(Q_{1\omega n} + 2\pi\frac{\Omega_i}{q_i}\theta_{k1}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left[\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\mathfrak{I}_{0\mu} + \mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\right]\right)\phi_{10\omega n}$$
$$= 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\mathfrak{I}_{0\mu}\left(\frac{h^+_{1\mu b\omega n}}{a^+_{\mu b}} - \frac{h^-_{1\mu b\omega n}}{a^-_{\mu b}}\right) + 2\pi\frac{\Omega_i}{q_i}\theta_{k1}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\left(\frac{h^+_{0\mu b\omega n}}{a^+_{\mu b}} - \frac{h^-_{0\mu b\omega n}}{a^-_{\mu b}}\right), \quad (113)$$

and at the second order,

$$\mathcal{L}^\pm_{0\mu b\omega n}\left(\frac{h^\pm_{2\mu b\omega n}}{a^\pm_{\mu b}}\right) + \mathcal{M}^\pm_{0\mu b\omega n}\Phi_{12\omega n} - \frac{q_i}{m_i}\mathcal{L}^\pm_{0\mu b\omega n}\left(\frac{\Phi_{12\omega n}}{a^\pm_{\mu b}}\right) + \mathcal{L}^\pm_{1\mu b\omega n}\left(\frac{h^\pm_{1\mu b\omega n}}{a^\pm_{\mu b}}\right) + \mathcal{M}^\pm_{1\mu b\omega n}\Phi_{11\omega n}$$
$$- \frac{q_i}{m_i}\mathcal{L}^\pm_{1\mu b\omega n}\left(\frac{\Phi_{11\omega n}}{a^\pm_{\mu b}}\right) + \mathcal{L}^\pm_{2\mu b\omega n}\left(\frac{h^\pm_{0\mu b\omega n}}{a^\pm_{\mu b}}\right) + \mathcal{M}^\pm_{2\mu b\omega n}\Phi_{10\omega n} - \frac{q_i}{m_i}\mathcal{L}^\pm_{2\mu b\omega n}\left(\frac{\Phi_{10\omega n}}{a^\pm_{\mu b}}\right) = 0, \quad (114)$$

$$\left(Q_{0\omega n} + 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\mathfrak{I}_{0\mu}\right)\phi_{12\omega n}$$
$$+\left(Q_{1\omega n} + 2\pi\frac{\Omega_i}{q_i}\theta_{k1}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left[\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\mathfrak{I}_{0\mu} + \mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\right]\right)\phi_{11\omega n}$$
$$+\left(Q_{2\omega n} + 2\pi\frac{\Omega_i}{q_i}\theta_{k2}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left[\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\mathfrak{I}_{0\mu} + \mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\right]\right.$$
$$+ 2\pi\frac{\Omega_i}{q_i}\frac{\theta^2_{k1}}{2}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left[\partial^2_{\theta_{k0}}\mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\mathfrak{I}_{0\mu} + \mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\partial^2_{\theta_{k0}}\mathfrak{I}_{0\mu}\right.$$
$$\left.\left. + 2\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\left\{\frac{1}{a^+_{\mu b}} - \frac{1}{a^-_{\mu b}}\right\}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\right]\right)\phi_{10\omega n} =$$

$$2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\mathfrak{I}_{0\mu}\left(\frac{h^+_{2\mu b\omega n}}{a^+_{\mu b}} - \frac{h^-_{2\mu b\omega n}}{a^-_{\mu b}}\right) + 2\pi\frac{\Omega_i}{q_i}\theta_{k1}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\left(\frac{h^+_{1\mu b\omega n}}{a^+_{\mu b}} - \frac{h^-_{1\mu b\omega n}}{a^-_{\mu b}}\right)$$
$$+ 2\pi\frac{\Omega_i}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\left(\theta_{k2}\partial_{\theta_{k0}}\mathfrak{I}_{0\mu} + \frac{1}{2}\theta^2_{k1}\partial^2_{\theta_{k0}}\mathfrak{I}_{0\mu}\right)\left(\frac{h^+_{0\mu b\omega n}}{a^+_{\mu b}} - \frac{h^-_{0\mu b\omega n}}{a^-_{\mu b}}\right). \quad (115)$$

We observe that (110), (112) and (114) have respectively the same structure as (65), (67), and (69). The only differences come from the potential function. Therefore, by solving (110), (112), and (114), we obtain the same results (73), (75), (77), (80), and (101), provided we replace $\phi_{1j\omega n}$ by $\Phi_{1j\omega n}$ for $j \in \{0,1,2\}$. From (73), (75), (77), (80), and (101), and observing that $h^\pm_{j\mu b\omega n}[\Phi_{1j\omega n}]$ are linear functionals of $\Phi_{1j\omega n}$ for $j \in \{0,1,2\}$, we obtain, after some algebra,

$$h^\pm_{0\mu b\omega n}[\Phi_{10\omega n}] = h^\pm_{0\mu b\omega n}[\mathfrak{I}_{0\mu}\phi_{10\omega n}], \quad (116)$$
$$h^\pm_{1\mu b\omega n}[\Phi_{11\omega n}] = h^\pm_{1\mu b\omega n}[\mathfrak{I}_{0\mu}\phi_{11\omega n}] + \theta_{k1}h^\pm_{0\mu b\omega n}[\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\phi_{10\omega n}], \quad (117)$$
$$h^\pm_{2\mu b\omega n}[\Phi_{12\omega n}] = h^\pm_{2\mu b\omega n}[\mathfrak{I}_{0\mu}\phi_{12\omega n}] + \theta_{k1}h^\pm_{1\mu b\omega n}[\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\phi_{11\omega n}]$$
$$+ \theta_{k2}h^\pm_{0\mu b\omega n}[\partial_{\theta_{k0}}\mathfrak{I}_{0\mu}\phi_{10\omega n}] + \frac{1}{2}\theta^2_{k1}h^\pm_{0\mu b\omega n}[\partial^2_{\theta_{k0}}\mathfrak{I}_{0\mu}\phi_{10\omega n}]. \quad (118)$$

Substituting (116) into (111), the zeroth-order integral equation becomes

$$\left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle \phi_{10\omega n} = 0, \quad \text{with} \quad \left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle = \left\langle Q^{\circ}_{\omega n}\right\rangle + \left\langle \mathcal{L}^{\circ}_{O\omega n}\right\rangle + \left\langle \mathcal{L}^{\circ}_{C\omega n}\right\rangle.$$

The operators $\langle Q^{\circ}_{\omega n}\rangle$, $\langle \mathcal{L}^{\circ}_{O\omega n}\rangle$, and $\langle \mathcal{L}^{\circ}_{C\omega n}\rangle$ are defined by

$$\left\langle Q^{\circ}_{\omega n}\right\rangle = Q_{0\omega n} + 2\pi \frac{\Omega_i}{q_i} \sum_{\mu b \in \mathscr{C}} \mathcal{A}_{\mu b} \mathfrak{I}_{0\mu} \left(\frac{1}{a^{+}_{\mu b}} - \frac{1}{a^{-}_{\mu b}} \right) \mathfrak{I}_{0\mu},$$

$$\left\langle \mathcal{L}^{\circ}_{O\omega n}\right\rangle = \sum_{\mu b \in O} \mathfrak{I}_{0\mu} \mathcal{L}^{\circ}_{O\mu b\omega n} \mathfrak{I}_{0\mu}, \quad \left\langle \mathcal{L}^{\circ}_{C\omega n}\right\rangle = \sum_{\mu b \in C} \mathfrak{I}_{0\mu} \mathcal{L}^{\circ}_{C\mu b\omega n} \mathfrak{I}_{0\mu}.$$

Substituting (116) and (117) into (113), the first-order equation becomes, after some algebra,

$$\left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle \phi_{11\omega n} + \left(\theta_{k1} \partial_{\theta_{k0}} \left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle + \omega_1 \partial_{\omega_0} \left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle - \frac{\mathrm{i}}{2} \frac{1}{nq'} \partial_x \theta_{k0} \partial^2_{\theta_{k0}} Q_{0\omega n} \right) \phi_{10\omega n} = 0.$$

Since the previous equation has the same structure as (76), all the conclusions inferred from solving (76) in Sec. IV C remain valid and identical. Substituting (116)-(118) into (115), the second-order equation becomes, after some algebra,

$$\Big(\partial_{\theta_{k0}} \left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle \left\{ \phi_{10\omega n} \Theta_{k2} + \partial_{\theta_{k0}} \phi_{10\omega n} \Theta^2_{k1} + g_1 \phi_{10\omega n} \Theta_{k1} \right\} + \frac{1}{2} \partial^2_{\theta_{k0}} \left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle \phi_{10\omega n} \Theta^2_{k1}$$
$$+ \left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle \phi_{12\omega n} + \omega_2 \partial_{\omega_0} \left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle \phi_{10\omega n} \Big) A_1 = 0.$$

Since the last equation has the same structure as (107), all the conclusions inferred from solving (107) in Sec. IV D also remain valid and identical. Particularly, we obtain the same Schrödinger equation (99) or Riccati equation (100). □

F. An algorithm for solving the eigenvalue problem

We now summarize the results of the zeroth-, first-, and second-order problems of the above asymptotic analysis in order to design an algorithm for solving the eigenvalue problem.

(a) First for any fixed value $q_* \in [q_{\min}, q_{\max}]$, we have to solve the eigenvalue problem associated to the one-dimensional integral operator of Proposition 7, i.e.,

$$\left\langle \mathcal{L}^{\circ}_{\mathscr{C}\omega n}\right\rangle \phi_{10\omega n} = 0, \tag{119}$$

for different values of the parameter θ_{k0} to reconstruct the local eigenfrequency $\omega_0(q_*, \theta_{k0})$. Knowing $\omega_0(q_*, \theta_{k0})$, we can determine the value of the turning point $\theta_{k0,T}$ for which $\partial_{\theta_{k0}} \omega_0(q_*, \theta_{k0,T}) = 0$. From the θ-symmetry of the equilibrium and given that the toroidal-ITG instability has maximum amplitude on the low-field side, we may expect $\theta_{k0,T}$ to be close to the origin, thus limiting our search.

(b) Once the turning point $\theta_{k0,T}$ is known for every value $q \in [q_{\min}, q_{\max}]$, we have to solve the eigenvalue problem (119) to reconstruct the local eigenfrequency $\omega_0(q, \theta_{k0,T})$ and get the eigenfunction $\phi_{10\omega n}$ which gives the slow poloidal θ-envelope of the eigenmode.

(c) Once the local eigenfrequency $\omega_0(q, \theta_{k0,T})$ is obtained, we can solve (numerically) the non-self-adjoint Schrödinger equation (99): this consists of finding $\mathscr{S}_n$, i.e., the set of $\omega \in \mathbb{C}$ such that the kernel of (99) is non-trivial or empty. The set $\mathscr{S}_n$ constitutes the point spectrum of our problem, while the associated eigenvectors give the slow radial q-envelope A_1 of the global eigenmode.

(d) Then, using the ballooning representation (36) (see Sec. III C 2), in which we replace $\widehat{\phi}_{\omega n}$ by $\phi_{10\omega n}$, and θ_k by $\theta_{k0,T} + \theta_{k1}$, or $\exp(in \int dq\, \theta_k)$ by $\exp(inq\theta_{k0,T}) A_1(q)$, we obtain the desired three-dimensional eigenmode $\phi(t, \mathbf{r})$.

Finally the heart of the problem is to solve an eigenvalue problem for the one-dimensional linear integral operator (119).

V. SPECTRAL ANALYSIS

In this section, we perform the spectral analysis for the one-dimensional non-self-adjoint Schrödinger-type operator and for the nested one-dimensional Fredholm-type integral operator with a nonlinear dependency of the eigenparameter.

A. The schrödinger-type radial envelope equation

We aim at giving some general spectral properties of the Schrödinger equation (99) and to present a particular resolution of (99) in the case of two closely spaced simple turning points under some additional assumptions on the local eigenfrequency ω_0. We consider the non-self-adjoint Schrödinger equation (99), recast as the boundary value problem with homogeneous Dirichlet conditions:

$$T(\omega)A_1 = 0, \qquad A_1(q_{\min}) = 0, \;\; A_1(q_{\max}) = 0. \tag{120}$$

Here, we use the following notation:

$$T(\omega) = \frac{\partial^2}{\partial q^2} - n^2 Q(q,\omega), \quad Q(q,\omega) = -\frac{\omega - \omega_0(q,\theta_{k0,T})}{\frac{1}{2}\partial^2_{\theta_{k0}}\omega_0(q,\theta_{k0,T})} \in \mathbb{C}, \tag{121}$$

where $q_{\max} = q(r_{\max})$ and $q_{\min} = q(r_{\min})$.

1. General case

Let us first give some general spectral properties of the non-self-adjoint boundary value problem (120), which can also be seen as a non-self-adjoint Sturm-Liouville problem. We have the following spectral theorem.

Theorem 1. *Let us assume that the complex radial functions* $q \longmapsto \omega_0(q,\theta_{k0,T})$ *and* $q \longmapsto \partial^2_{\theta_{k0}}\omega_0(q,\theta_{k0,T})$ *are such that*

$$\frac{1}{\partial^2_{\theta_{k0}}\omega_0}, \; \frac{\omega_0}{\partial^2_{\theta_{k0}}\omega_0} \in L^1([q_{\min}, q_{\max}];\mathbb{C}).$$

Let Ω *be any open connected subset of* $\mathbb{C}$ *and let the toroidal number* $n \in \mathbb{Z}$ *be fixed. Then either*

(i) $T^{-1}(\omega)$ *exists for no* $\omega \in \Omega$ *and* $\operatorname{Ker} T \neq \{0\}$, $\forall \omega \in \Omega$.
(ii) $T^{-1}(\omega)$ *exists for all* $\omega \in \Omega$ *and* $\operatorname{Ker} T = \{0\}$, $\forall \omega \in \Omega$.
(iii) $T^{-1}(\omega)$ *exists for all* $\omega \in \Omega\backslash\mathscr{S}_n$ *where* $\mathscr{S}_n$ *is a discrete subset of* Ω *constituted of an (infinite or finite) countable number of isolated points (i.e., a set which has no accumulation point in* Ω *and contains a finite number of singular points — poles — in each compact subset of* Ω*). In this case* $\omega \longmapsto T^{-1}(\omega)$ *is a meromorphic operator-valued function in* Ω*, analytic in* $\Omega\backslash\mathscr{S}_n$*, and the residues at the poles are finite rank operators such that* $\operatorname{Ker} T(\omega) \neq \{0\}$ *for* $\omega \in \mathscr{S}_n$*. Therefore, if* $\omega \in \mathscr{S}_n$*, the boundary value problem (120) has at most two linearly independent solutions which are not zero almost everywhere. More precisely, if* $\omega \in \mathscr{S}_n$ *and* $A_1 \in \operatorname{Ker} T(\omega)$ *then* $A_1 \in W^{2,1} \cap W^{1,\infty}([q_{\min}, q_{\max}];\mathbb{C})$ *and* $\dim \operatorname{Ker} T(\omega) = 1$.

Proof. Under assumptions of Theorem 1, items (i)–(iii) follow from Lemma 3.2.1 to Lemma 3.2.4 of Chapter 3 of Ref. 107. Particularly, the fact that $\dim \operatorname{Ker} T(\omega) = 1$ comes from Lemma 3.2.2 of Chapter 3 of Ref. 107 and the homogeneous Dirichlet boundary condition of the problem (120). In addition, the meromorphic property of the operator-valued function $\omega \longmapsto T^{-1}(\omega)$ and the finite rank residues can be deduced from the Laurent series expansion of the Green function of the non-self-adjoint boundary value problem (120) given by Theorem 3.8.1 of Chapter 3 of Ref. 107. □

Remark 15. Contrary to the spectral theory of self-adjoint Schrödinger (or Sturm-Liouville) operators, which is nowadays well understood and established, the non-self-adjoint theory (with *complex-valued potential) is in its infancy and no general results concerning the asymptotic behaviour (for large n) of the spectrum of (120) are known.*[26,35]

Remark 16. In the case where $\partial^2_{\theta_{k0}}\omega_0 \in \mathbb{R}^-_*$ (negative real), the spectral analysis of the non-self-adjoint Schrödinger (or Sturm-Liouville) operator (120) has been performed in Refs. 18, 85, and *107. Let us also point out the pioneering works of Sims*[93,32] *and Glazman*[45] *(§35 & §66) concerning the spectral analysis of Schrödinger operators with complex potentials.*

2. Case of two closely spaced simple turning points

Let us first introduce some notation. We set $z = q - q_0$ with $q_0 = q(r_0)$ and where r_0 is a reference rational magnetic flux surface between $r_{\min}$ and $r_{\max}$, say at the middle radius. We use the notation $\partial^k_q\omega^0_0 = \partial^k_q\omega_0(q,\theta_{k0,T})_{|q=q_0}$, and $\partial^k_q\partial^2_{\theta_{k0}}\omega^0_0 = \partial^k_q\partial^2_{\theta_{k0}}\omega_0(q,\theta_{k0,T})_{|q=q_0}$, for $0 \le k \le 3$. In the case of two closely spaced simple turning points we have the following

Theorem 2. *Let us assume that*

$$\left.\begin{aligned}&|z/q_0| \ll 1, \quad \omega_0, \partial^2_{\theta_{k0}}\omega_0 \in \mathscr{C}^3([q_{\min}, q_{\max}], \mathbb{C}),\\ &\partial_q\omega^0_0 = 0, \quad \omega^0_0 \neq 0, \quad \partial^2_q\omega^0_0 \neq 0, \quad \partial^2_{\theta_{k0}}\omega^0_0 \neq 0, \quad \left|q^k_0\frac{\partial^k_q\partial^2_{\theta_{k0}}\omega^0_0}{\partial^2_{\theta_{k0}}\omega^0_0}\right| = O(|z/q_0|^{3-k}), k = 1, 2.\end{aligned}\right\} \tag{122}$$

Then the eigenvalue problem associated to the non-self-adjoint Schrödinger boundary value problem (120) reduces to the solution of the algebraic dispersion equation $\mathbb{D}(\omega) = 0$, where

$$\mathbb{D}(\omega) = U(\mathfrak{a}(n,\omega^0_0,\omega),\mathfrak{z}_{\min})V(\mathfrak{a}(n,\omega^0_0,\omega),\mathfrak{z}_{\max}) - U(\mathfrak{a}(n,\omega^0_0,\omega),\mathfrak{z}_{\max})V(\mathfrak{a}(n,\omega^0_0,\omega),\mathfrak{z}_{\min}). \tag{123}$$

Here, the functions $U(\cdot,\cdot)$ and $V(\cdot,\cdot)$ are the linearly independent parabolic cylinder functions. We also use the following definitions

$$\mathfrak{z}_{\min} = \frac{q_{\min} - q_0}{z_0},\ \mathfrak{z}_{\max} = \frac{q_{\max} - q_0}{z_0},\ \ \mathfrak{a} = \mathfrak{a}(n,\omega^0_0,\omega) = \frac{n(\omega^0_0 - \omega)}{(2\partial^2_q\omega^0_0\partial^2_{\theta_{k0}}\omega^0_0)^{1/2}},\ \ z_0 = \left(\frac{1}{2n^2}\frac{\partial^2_{\theta_{k0}}\omega^0_0}{\partial^2_q\omega^0_0}\right)^{1/4}. \tag{124}$$

Proof. We first set $\widetilde{Q}(z) = Q(q_0 + z)$ and $\widetilde{A}_1(z) = A_1(q_0 + z)$. Using assumptions (122), and the Taylor series expansion we obtain

$$\widetilde{Q}(z) = \frac{\omega^0_0 - \omega + \frac{z^2}{2}\partial^2_q\omega^0_0}{\partial^2_{\theta_{k0}}\omega^0_0} + O(|z/q_0|^3).$$

Then, the non-self-adjoint boundary value problem (120) is approximated to third order, for small z/q_0, by

$$(\partial^2_z - \widetilde{Q})\widetilde{A}_1 = 0, \quad \widetilde{A}_1(z_{\min}) = 0,\ \widetilde{A}_1(z_{\max}) = 0,$$

where $z_{\min} = q_{\min} - q_0$ and $z_{\max} = q_{\max} - q_0$. In the previous equation, making the change of variables $z = z_0\mathfrak{z}$, where $z_0 \neq 0$ is a complex constant given by (124), and setting $A(\mathfrak{z}) = \widetilde{A}_1(z)$, we finally obtain the following Weber equation:

$$\partial^2_{\mathfrak{z}}A - \left(\mathfrak{a} + \frac{\mathfrak{z}^2}{4}\right)A = 0, \quad A(\mathfrak{z}_{\min}) = 0,\ A(\mathfrak{z}_{\max}) = 0, \tag{125}$$

with $\mathfrak{z}_{\min}$, $\mathfrak{z}_{\max}$, and $\mathfrak{a}$ given by (124). It is well known[1] that a fundamental system of solutions for the Weber equation (125) is formed by the two linearly independent parabolic cylinder functions $U(\mathfrak{a},\mathfrak{z})$ and $V(\mathfrak{a},\mathfrak{z})$. Therefore a solution of the boundary value problem (125) is given by a linear combination of the functions $U(\mathfrak{a},\mathfrak{z})$ and $V(\mathfrak{a},\mathfrak{z})$ for specific values of the global eigenvalue ω such that the boundary conditions $A(\mathfrak{z}_{\min}) = 0$ and $A(\mathfrak{z}_{\max}) = 0$ are satisfied. Now, using the Wronskian $W\{U(\mathfrak{a},\mathfrak{z}), V(\mathfrak{a},\mathfrak{z})\} = \sqrt{2/\pi}$ (see Ref. 1), and Lemma 3.2.2 of Chapter 3 of Ref. 107, we obtain that the values ω, for which the boundary conditions of equation (125) are satisfied, are solutions of the dispersion or characteristic equation $\mathbb{D}(\omega) = 0$, given by equation (123) of Theorem 2. □

Remark 17. In order to find the solutions of the dispersion equation $\mathbb{D}(\omega) = 0$, we may use the asymptotic approximations (see Ref. 1) of $U(\mathfrak{a},\mathfrak{z})$ and $V(\mathfrak{a},\mathfrak{z})$ since $|\mathfrak{z}_{\min}| = O(n^{1/2})$, $|\mathfrak{z}_{\max}| = O(n^{1/2})$ (using (124)), and $|\mathfrak{a}| = O(1)$ (see Remark 18) or perform directly a numerical resolution.

Remark 18. Actually, we may expect that $\omega - \omega_0^0 = O(1/n)$. Indeed, let us choose a symmetric approximation of the functions $U(\mathfrak{a},\mathfrak{z})$ and $V(\mathfrak{a},\mathfrak{z})$ given, respectively, by $\exp(-\mathfrak{z}^2/4)$ and $\exp(\mathfrak{z}^2/4)$. We then take $A(\mathfrak{z}) \simeq \exp(-s\mathfrak{z}^2/4)$ where $s = \mathrm{sign}(\mathrm{Re}\,(\partial_q^2\omega_0^0/\partial_{\theta_{k0}}^2\omega_0^0)^{1/2})$ such that the radial envelope approximation $A_1(q) \simeq \exp(-sn(\partial_q^2\omega_0^0/\partial_{\theta_{k0}}^2\omega_0^0)^{1/2}/(2\sqrt{2})(q-q_0)^2)$ has its modulus decreasing when $|q - q_0|$ increases. Substituting the approximation $A(\mathfrak{z}) \simeq \exp(-s\mathfrak{z}^2/4)$ into (125), we obtain

$$\omega = \omega_0^0 + \frac{1}{2\sqrt{2}}\frac{s}{n}(\partial_q^2\omega_0^0\partial_{\theta_{k0}}^2\omega_0^0)^{1/2},$$

which means that $\omega - \omega_0^0 = O(1/n)$ and thus using (124), $|\mathfrak{a}| = O(1)$. Let us now estimate how close the two simple turning points $\{q_{T_i}\}_{i=1,2}$ are. Since $\omega - \omega_0 = O(\epsilon^{2\gamma})$ with $\gamma < 1/2$, using the following Taylor expansion:

$$\omega - \omega_0(q_{T_i}) = \omega - \omega_0^0 + \omega_0(q_0) - \omega_0(q_{T_i}) = O(1/n) + (q_0 - q_{T_i})\partial_q\omega_0(q_{T_i}) + O((q_0 - q_{T_i})^2),$$

we find that the two simple turning points are separated by a distance Δq smaller than or equal to $n^{-\nu}a$ with $\nu = 2\gamma < 1$, if $\partial_q\omega_0(q_{T_i}) \neq 0$. This estimation must be compared with the radial extension a priori estimate of the global eigenmode from the phase factor $n\int dq\,\theta_{k1} = O(1)$ of the eikonal form (34). Since $|\theta_{k1}| = O(\epsilon^\gamma)$, we get $\Delta q = O(n^{-\sigma}a)$ with $\sigma = 1 - \gamma > 1/2$. These estimations are in good agreement and equal if we choose $\gamma = 1/3$ ($\nu = \sigma = 2/3$). Taking $\gamma = 1/3$ leads to a radial extension of the eigenmode of order $n^{-2/3}a$.

B. The nested Fredholm-type integral operator

In this section we discuss the solution of the linear homogeneous Fredholm's integral equation $\langle\mathcal{L}^\circ_{\mathscr{C}\omega n}\rangle\,\phi = 0$, whose kernel depends nonlinearly on the eigenvalue parameter ω_0. Although the theory of linear Fredholm integral equations with linear eigenparameter dependence is well known;[37,63,84,19,62,65,64,66,94,95,83] in the case of a nonlinear eigenvalue parameter dependence, and more generally for nonlinear eigenvalue problems, the theory is far from being complete despite a lot of analytical[97,68,52,53,36,103,104,86,87,25,34] and numerical[91,57,51,101] developments. In order to use the theory of the linear Fredholm's integral equation with linear eigenparameter dependence, we may, by means of some transformations, increase the dimension of the space of unknowns and convert the nonlinear eigenvalue problem into a generalized (linear) eigenvalue problem. Using appropriate changes of variables and taking into account only open contours, we are able to linearize the nonlinear eigenvalue problem $\langle\mathcal{L}^\circ_{\mathscr{C}\omega n}\rangle\,\phi = 0$ (see Remark 23). For closed contours, nonlinear terms with respect to the eigenfrequency still remain, and it seems difficult to get rid of them. Another point of view for considering the problem $\langle\mathcal{L}^\circ_{\mathscr{C}\omega n}\rangle\,\phi = 0$ is the perturbation theory of linear operators depending continuously on a parameter, for instance, the eigenfrequency ω_0.[31,69] We first give some properties of the integral operator, next consider the case of only open contours, and finally deal with the case of both open and closed contours.

1. *Basic properties of the integral operator*

The integral equation $\langle\mathcal{L}^\circ_{\mathscr{C}\omega n}\rangle\,\phi = 0$, satisfied by the electrical potential $\phi = \phi_{10\omega n}$, can be rewritten as

$$\phi(\theta) = \int_{-\infty}^{\infty} d\eta\,\mathbb{K}(\theta,\eta;\omega_0)\phi(\eta) = \mathrm{Op}(\mathbb{K})\phi(\theta) = \big(\mathrm{Op}(\mathbb{K}_O) + \mathrm{Op}(\mathbb{K}_C)\big)\phi(\theta), \tag{126}$$

where

$$\mathbb{K}(\theta,\eta;\omega_0) = \mathbb{K}_O(\theta,\eta;\omega_0) + \mathbb{K}_C(\theta,\eta;\omega_0) = \sum_{\mu b\in O}\mathbb{K}_{O\mu b}(\theta,\eta;\omega_0) + \sum_{\mu b\in C}\mathbb{K}_{C\mu b}(\theta,\eta;\omega_0).$$

Here, the kernels $\mathbb{K}_{O\mu b}$ and $\mathbb{K}_{C\mu b}$ are defined by

$$\mathbb{K}_{O\mu b}(\theta,\eta;\omega_0) = -\mathrm{i}2\pi\frac{\Omega_i}{q_i}\frac{\mathcal{A}_{\mu b}}{Q(\theta)}\frac{\mathfrak{J}_{0\mu}(\theta)}{a_{\mu b}(\theta)}\frac{\mathfrak{J}_{0\mu}(\eta)}{a_{\mu b}(\eta)}\left(\frac{e}{T_{e0}}\frac{qR}{b_\varphi}\Omega_{*\mu b}(\omega_0)a^2_{\mu b}\right)(\eta)\times \exp\left(\mathrm{i}\,\mathrm{sign}(\eta-\theta)\mathcal{I}_{\mu b}(\theta,\eta;\omega_0)\right), \quad (127)$$

$$\mathbb{K}_{C\mu b}(\theta,\eta;\omega_0) = \sum_{\ell\in\mathbb{Z}} 2\pi\frac{\Omega_i}{q_i}\frac{\mathcal{A}_{\mu b}}{Q(\theta)}\frac{\mathfrak{J}_{0\mu}(\theta)}{a_{\mu b}(\theta)}\frac{\mathfrak{J}_{0\mu}(\eta)}{a_{\mu b}(\eta)}\frac{2\mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}]}(\theta)}{\sin\mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b};\omega_0)}\left(\frac{e}{T_{e0}}\frac{qR}{b_\varphi}\Omega_{*\mu b}(\omega_0)a^2_{\mu b}\right)(\eta) \left\{\cos\mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\eta;\omega_0)\cos\mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\theta;\omega_0)\mathbb{1}_{[\theta,\theta^{+,\ell}_{L\mu b}]}(\eta) + \cos\mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\eta;\omega_0)\cos\mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\theta;\omega_0)\mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta]}(\eta)\right\}, \quad (128)$$

where

$$Q(\theta) = Q^\circ(\theta) + 4\pi\frac{\Omega_i(\theta)}{q_i}\sum_{\mu b\in\mathscr{C}}\mathcal{A}_{\mu b}\frac{\mathfrak{J}^2_{0\mu}(\theta)}{a_{\mu b}(\theta)},$$

$$Q^\circ(\theta) = \frac{e\tau n_{i0}}{k_B T_{i0}} + \frac{n_{i0}\,n^2}{(\Omega_i B)(\theta)}\frac{q^2}{r^2}\left[1+s^2(\theta-\theta_{k0})^2\right],$$

$$\mathfrak{J}_{0\mu}(\theta) = J_0\left(\frac{|n|q}{r}\frac{v_\perp}{\Omega_i(\theta)}\sqrt{1+s^2(\theta-\theta_{k0})^2}\right),$$

$$\mathcal{I}_{\mu b}(\theta,\eta;\omega_0) = \int_\theta^\eta d\widehat{\eta}\,[\omega_0-\omega_{d\mu b}(\widehat{\eta})+\mathrm{i}\omega_{\diamond\mu b}(\widehat{\eta})]\left(\frac{qR}{b_\varphi a_{\mu b}}\right)(\widehat{\eta}),$$

$$\Omega^\pm_{*\mu b}(\eta;\omega_0) = \left\{\omega_{*\mu b}(\eta)a^2_{\mu b}(\eta) + \frac{q_i}{m_i}\frac{T_{e0}}{e}[\omega_0-\omega_{d\mu b}(\eta)+\mathrm{i}\omega_{\diamond\mu b}(\eta)]\right\}\frac{1}{a^2_{\mu b}(\eta)},$$

$$\omega_{d\mu b}(\eta) = \frac{nb_\varphi}{R(\eta)}\left[\frac{\mu}{q_i}+\left(\frac{a^2_{\mu b}}{\Omega_i}\right)(\eta)\right]\left[q'(\eta-\theta_{k0})\sin\eta + \mathfrak{q}(\eta)\frac{\cos\eta}{r}\right],$$

$$\omega_{*\mu b}(\eta) = \frac{nb_\varphi T_{e0}}{e(Ba_{\mu b})(\eta)}\left\{\frac{1}{r}\left[(\mathfrak{q}\partial_x a_{\mu b})(\eta) - q'(\eta-\theta_{k0})\partial_\eta a_{\mu b}(\eta)\right] + \left[q'(\eta-\theta_{k0})\sin\eta+\mathfrak{q}(\eta)\frac{\cos\eta}{r}\right]\left(\frac{a_{\mu b}}{R}\right)(\eta)\right\},$$

$$\omega_{\diamond\mu b}(\eta) = -2\left[\sin\eta\,\partial_x a_{\mu b}(\eta) - \frac{\cos\eta}{r}\partial_\eta a_{\mu b}(\eta)\right]\left(\frac{a_{\mu b}}{\Omega_i}\frac{b_\varphi}{R}\right)(\eta),$$

$$a_{\mu b}(\theta) = a^\circ_{\mu b}\sqrt{1+\Lambda_{\mu b}(\cos\theta-1)}, \quad \text{for } \theta\in\,]-\pi,\pi[, \text{ if } (\mu,b)\in O, \quad \text{or } \theta\in\,]-\theta_{L\mu b},\theta_{L\mu b}[, \text{ if } (\mu,b)\in C,$$

$$\theta^{\pm,\ell}_{L\mu b} = \pm\theta_{L\mu b}+2\pi\ell,\ \ell\in\mathbb{Z}, \quad \theta_{L\mu b} = \left|\arccos\left(1-\Lambda_{\mu b}{}^{-1}\right)\right|,$$

$$\Lambda_{\mu b} = \frac{2\mu B_0}{m_i a^{\circ 2}_{\mu b}}\frac{r}{R_0}.$$

Here, the quantities $\Omega_i(\theta) = q_i B(\theta)/m_i$, $R(\theta) = R_0 + r\cos\theta$, $B(\theta) = B_0(1+r/R_0\cos\theta)^{-1}\sqrt{1+r^2/[qR(\theta)]^2} = B_0(1-r/R_0\cos\theta)+O(\epsilon_a^2)$, $b_\varphi = 1/\sqrt{1+r^2/[qR(\theta)]^2} = 1+O(\epsilon_a^2)$, $\mathfrak{q}(\theta) = q[r^2+q^2R(\theta)^2]/[qR(\theta)]^2 = q(r)+O(\epsilon_a^2)$, $s = q'r/q$, $a^\circ_{\mu b} = a^\circ_{\mu b}(r)$, $n_{i0} = n_{i0}(r)$, $T_{i0} = T_{i0}(r)$, and $T_{e0} = T_{e0}(r)$ are given quantities. We assume that n_{i0}, $T_{i0} > 0$, and that $n_{i0}(r)$, $T_{i0}(r)$, $q(r)$, and $s(r)$ are regular enough typically in the space $\mathscr{C}^m_b([r_{\min},r_{\max}])$ with $m\geq 1$. Let us now make two important remarks.

Remark 19. First we can straightforwardly show that (i) the kernel $\mathbb{K}(\theta,\eta;\omega_0)$ is 2π-periodic in θ_{k0}; that is if we add 2π to θ_{k0} and make the change of variables $\eta=\eta'+2\pi$ and $\theta=\theta'+2\pi$, *thanks to θ-periodicity of equilibrium quantities, we obtain the same kernel $\mathbb{K}$ in the variables η' and θ'; (ii) the spectrum of the integral operator (126) is symmetric with respect to the sign of the toroidal number n; that is, if we change n into $-n$, then we obtain the same integral operator (126) where $\omega_0 = \omega_0^{\Re} + \mathrm{i}\omega_0^{\Im}$ and ϕ are changed, respectively, into $-\omega_0^{\Re} + \mathrm{i}\omega_0^{\Im}$ and $\phi^\star$ in (126).*

Remark 20. If we choose the approximation $B = B_0R_0/R + O(\epsilon_a^2)$, then we get $\partial_\eta B = Br\sin\eta/R + O(\epsilon_a^2)$ and $\partial_x B = -B\cos\eta/R + O(\epsilon_a^2)$. Using this approximation and the approximated equilibrium contours (28), we obtain the formula

$$\frac{e}{T_{e0}}\Omega_{*\mu b}a_{\mu b}^2 = \frac{q_i}{m_i}(\omega_0 + \mathrm{i}\omega_{\diamond\mu b}) + \frac{nb_\varphi}{B}\frac{\mathfrak{q}}{r}\frac{d}{dr}\left(\frac{a_{\mu b}^{\circ 2}}{2} + \frac{\mu B(r,0)}{m_i}\right) \tag{129}$$

$$= \frac{q_i}{m_i}\omega_0 + \frac{n\mathfrak{q}b_\varphi}{rB}\left(1 - \mathrm{i}\frac{2r}{n\mathfrak{q}R}\sin\eta\right)\frac{d}{dr}\left(\frac{a_{\mu b}^{\circ 2}}{2} + \frac{\mu B(r,0)}{m_i}\right) - \mathrm{i}\frac{2b_\varphi}{R^2B}\frac{\mu B}{m_i}\sin 2\eta$$

whose r.h.s. is bounded with respect to the η-variable. Without this approximation $(e/T_{e0})\Omega_{*\mu b}a_{\mu b}^2$ is linear in η and thus unbounded.

We now intend to study the spectral properties of the operator $I - \mathrm{Op}(\mathbb{K})$ defined by (126). For this, we will state few lemmas, propositions, and theorems. Let us begin with the following lemma.

Lemma 1. Let us suppose that $\{a_{\mu b}^\circ\}_{\mu b\in\mathscr{C}} \in \mathscr{C}_b^1([r_{\min}, r_{\max}])$ are bounded from below and above, i.e., there exist two constants $a_{\min}^\circ > 0$ and $a_{\max}^\circ > 0$ such that $a_{\min}^\circ \le a_{\mu b}^\circ \le a_{\max}^\circ$ for all $(\mu, b) \in \mathscr{C}$. Then,

$$\frac{1}{a_{\mu b}} \in L^\gamma_{\mathrm{loc}}(\mathbb{R}), \quad \gamma \in (0,2), \quad \forall(\mu,b) \in \mathscr{C}, \tag{130}$$

$$\int_{\theta^{-,\ell}_{L\mu b}}^{\theta^{+,\ell}_{L\mu b}} d\eta \left(\frac{qR}{b_\varphi}\frac{\omega_{\diamond\mu b}}{a_{\mu b}}\right)(\eta) = 0, \quad \forall(\mu,b) \in C, \ \forall\ell \in \mathbb{Z}, \tag{131}$$

$$\int_{\eta_0}^{\eta_0+2\pi} d\eta \left(\frac{qR}{b_\varphi}\frac{\omega_{\diamond\mu b}}{a_{\mu b}}\right)(\eta) = 0, \quad \forall(\mu,b) \in O, \quad \forall\eta_0 \in \mathbb{R}, \tag{132}$$

and

$$\left|\exp\left(-\operatorname{sign}(\eta' - \eta)\int_\eta^{\eta'} d\theta \left(\frac{qR}{b_\varphi}\frac{\omega_{\diamond\mu b}}{a_{\mu b}}\right)(\theta)\right)\right| \le C < \infty, \quad \forall\eta,\eta' \in \mathbb{R}, \quad \forall(\mu,b) \in O. \tag{133}$$

Proof. Since we have supposed that $\{a_{\mu b}^\circ\}_{\mu b\in\mathscr{C}}$ are continuous with respect to the r-variable, bounded from below and above we get $2r_{\min}\mu_{\min}B_0/(m_i a_{\max}^{\circ 2}R_0) \le \Lambda_{\mu b} \le 2r_{\max}\mu_{\max}B_0/(m_i a_{\min}^{\circ 2}R_0)$. As a consequence, we obviously have, for all fixed $r \in [r_{\min}, r_{\max}]$ and for all $(\mu, b) \in O$, that $1/a_{\mu b} \in L^1_{\mathrm{loc}}(\mathbb{R}_\theta)$. The only singular points are the limit angles $\theta^{\pm,\ell}_{L\mu b}$ where $a_{\mu b}$ vanishes, for all $(\mu, b) \in C$. For all $(\mu, b) \in C$ and for all fixed $r \in [r_{\min}, r_{\max}]$, in the neighborhood $\mathcal{V}(\theta^{\pm,\ell}_{L\mu b})$ of $\theta^{\pm,\ell}_{L\mu b}$, the closed contour $a_{\mu b}$ behaves as

$$\frac{1}{a_{\mu b}} \underset{\theta\simeq\theta^{\pm,\ell}_{L\mu b}}{=} \frac{1}{a_{\mu b}^\circ}\left(\sqrt{2\Lambda_{\mu b} - 1}(\pm\theta^{\pm,\ell}_{L\mu b} \mp \theta) + O\left((\pm\theta^{\pm,\ell}_{L\mu b} \mp \theta)^2\right)\right)^{-1/2}, \tag{134}$$

which is an integrable algebraic singularity in θ provided that $\Lambda_{\mu b} > 1/2$. Hence $1/a_{\mu b} \in L^\gamma_{\mathrm{loc}}(\mathbb{R}_\theta)$ for $0 < \gamma < 2$, and for all $(\mu, b) \in C$ (130) is stated. Let us now look at the term $qR\omega_{\diamond\mu b}/(a_{\mu b}b_\varphi)$, which can be written as

$$\left(\frac{qR}{b_\varphi}\frac{\omega_{\diamond\mu b}}{a_{\mu b}}\right)(\eta) = -\frac{2q}{\Omega_i}\left[\sin\eta\,\partial_x a_{\mu b}(\eta) - \frac{\cos\eta}{r}\partial_\eta a_{\mu b}(\eta)\right]$$

$$= -\frac{q}{\Omega_i}\frac{\sin\eta}{a_{\mu b}(\eta)}\partial_x\left(a_{\mu b}^2(\eta) + \frac{r\mu B_0}{R_0 m_i/2}\cos\eta\right),$$
$$= -\frac{q}{\Omega_i}\frac{\sin\eta}{a_{\mu b}(\eta)}\left(\partial_x a_{\mu b}^{\circ 2} + \frac{\mu B_0}{R_0 m_i/2}(2\cos\eta - 1)\right).$$

From the previous equation, for all $(\mu,b) \in C$, we observe that $qR\omega_{\diamond\mu b}/(a_{\mu b}b_\varphi)$ is odd with respect to the η-variable and thus, by periodicity, we get (131). Now for all $(\mu,b) \in O$, we observe that $qR\omega_{\diamond\mu b}/(a_{\mu b}b_\varphi)$ is periodic with a period of at most 2π. Moreover, using the change of variables in η given by $u = a_{\mu b}^{\circ 2} + \frac{r\mu B_0}{R_0 m_i/2}(\cos\eta - 1)$, we obtain

$$\int_\eta^{\eta'} d\theta\left(\frac{qR}{b_\varphi}\frac{\omega_{\diamond\mu b}}{a_{\mu b}}\right)(\theta) = -\frac{2q}{\Omega_i}\frac{R_0}{r}\frac{m_i a_{\mu b}^{\circ 2}/2}{\mu B_0}$$
$$\left[\left(a_{\mu b}^{\circ 2} + \frac{r\mu B_0}{R_0 m_i/2}(\cos\eta' - 1)\right)^{1/2}\left(\frac{1}{3r} - 2\frac{\partial_x a_{\mu b}^\circ}{a_{\mu b}^\circ} - \frac{2}{3R_0}\frac{\mu B_0}{m_i a_{\mu b}^\circ/2}(\cos\eta' - 1)\right)\right.$$
$$\left.-\left(a_{\mu b}^{\circ 2} + \frac{r\mu B_0}{R_0 m_i/2}(\cos\eta - 1)\right)^{1/2}\left(\frac{1}{3r} - 2\frac{\partial_x a_{\mu b}^\circ}{a_{\mu b}^\circ} - \frac{2}{3R_0}\frac{\mu B_0}{m_i a_{\mu b}^\circ/2}(\cos\eta - 1)\right)\right] = g(\eta') - g(\eta),$$

where $\eta \mapsto g(\eta)$ is a 2π-periodic function. Therefore, we get (132) from which we deduce (133). □

2. Case of only open contours

We first consider the case of only open contours. The case of only open contours is interesting in itself, since with only open contours, we can take into account the behaviour of both passing particles (open trajectories) and some trapped particles (closed trajectories). The main theorems of this section are Theorem 3 (for the spectrum in $\mathbb{C}^+$) and Theorem 4 (for the spectrum in $\mathbb{C}$), which are established by using Proposition 8 and Proposition 9.

We start with a proposition stating that the operator $\mathrm{Op}(\mathbb{K}_O)$ is a bounded linear Hilbert-Schmidt operator in the Hilbert space $L^2(\mathbb{R})$. Let us recall that if a Hilbert-Schmidt operator is compact — and thus behaves as a finite-dimensional operator — the converse is false. In particular, the singular values of a Hilbert-Schmidt operator are square summable. Let us remark that for open contours the parameter μ is non-negative. Let us also introduce two integers δ and σ, such that

$$\delta = \begin{cases} 0 & \text{if there exists a } \mu \text{ such that } \mu = 0,\\ 1 & \text{if } \mu > 0 \text{ for all } \mu,\end{cases}$$

and

$$\sigma = \begin{cases} 0 & \text{if the approximation (129) of Remark 20 is assumed,}\\ 1 & \text{if the approximation (129) of Remark 10 is not assumed.}\end{cases}$$

Proposition 8. Under the assumptions of Lemma 1, for all values of ω_0 in the upper half-complex plane ($\mathrm{Im}\,\omega_0 > 0$), the operator $\mathrm{Op}(\mathbb{K}_O)$ is a Hilbert-Schmidt operator and thus compact on $L^2(\mathbb{R})$. Defining $\mathbb{C}^+ = \mathbb{C}\backslash\{\omega_0 \in \mathbb{C}\,|\,\mathrm{Im}\,\omega_0 \le 0\}$, we find that $\mathrm{Op}(\mathbb{K}_O)(\omega_0) : \mathbb{C}^+ \longmapsto \mathscr{L}(L^2)$ is an analytic operator-valued function such that $\mathrm{Op}(\mathbb{K}_O)(\omega_0)$ is compact for each $\omega_0 \in \mathbb{C}^+$.

Proof. Let us show that the operator $\mathrm{Op}(\mathbb{K}_O)$ belongs to the class of Hilbert-Schmidt operators for all values of ω_0 in the upper half-complex plane ($\mathrm{Im}\,\omega_0 > 0$), in other words that the kernel $\mathbb{K}_O \in L^2(\mathbb{R}\times\mathbb{R})$, i.e.,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathbb{K}_O(\theta,\eta;\omega_0)|^2 d\theta d\eta < \infty, \ \forall\omega_0 \text{ such that } \mathrm{Im}\,\omega_0 > 0.$$

From definition (127), taking into account Remark 20, and using (133) of Lemma 1, we can recast $\mathbb{K}_{O\mu b}$ as

$$\mathbb{K}_{O\mu b}(\theta,\eta;\omega_0) = \mathcal{B}_{O\mu b}(\theta,\eta;\omega)\frac{(1+|\eta|)^\sigma}{Q(\theta)}\frac{\mathfrak{I}_{0\mu}(\theta)}{a_{\mu b}(\theta)}\frac{\mathfrak{I}_{0\mu}(\eta)}{a_{\mu b}(\eta)}\exp\left(\mathrm{i}\omega_0\left|\int_\theta^\eta\left(\frac{qR}{b_\varphi a_{\mu b}}\right)(\eta')d\eta'\right|\right),$$

where $\mathcal{B}_{O\mu b}(\theta,\eta;\omega_0) \in L^\infty(\mathbb{R}_\eta \times \mathbb{R}_\theta)$. Using the property $0 < a_{\mu b} < \infty$ for $(\mu, b) \in O$, and (133) we obtain that, for all $(\mu, b) \in O$, there exists a constant $C = C(\mu, b, \omega_0)$ such that

$$|\mathbb{K}_{O\mu b}(\theta,\eta;\omega_0)|^2 \le C\frac{(1+|\eta|)^{2\sigma}}{(1+|\theta|^2)^2}|\mathfrak{J}_{0\mu}(\theta)|^2|\mathfrak{J}_{0\mu}(\eta)|^2 \exp\left(2\mathrm{i}\omega_0\left|\int_\theta^\eta \left(\frac{qR}{b_\varphi a_{\mu b}}\right)(\eta')d\eta'\right|\right). \quad (135)$$

Using the change of variables $\tilde{\eta} = \eta - \theta$ into (135), and the asymptotic property $\mathfrak{J}_{0\mu}(\eta) = O(1/\sqrt{|\eta|})$ as $\eta \to \pm\infty$, we then obtain that there exist two constants $C_1 = C_1(\mu, b, \omega_0)$ and $C_2 = C_2(\mu, b)$ such that

$$|\mathbb{K}_{O\mu b}(\theta,\eta;\omega_0)|^2 \le C_1(1+|\theta|^2)^{-2-\delta/2}(1+|\tilde{\eta}+\theta|^2)^{\sigma-\delta/2}\exp\left(-C_2\mathrm{Im}\ \omega_0|\tilde{\eta}|\right).$$

Hence $|\mathbb{K}_O(\theta,\eta;\omega_0)|^2 \in L^1(\mathbb{R}_\eta \times \mathbb{R}_\theta)$ as long as $\mathrm{Im}\,\omega_0 > 0$, which concludes the proof. From expression (127), analyticity of $\mathrm{Op}(\mathbb{K}_O)(\omega_0)$ with respect to ω_0 is obvious since functionals of ω_0 involve only products of polynomials and exponential of ω_0. □

Remark 21. The operator $\mathrm{Op}(\mathbb{K}_O)$ *belongs to the trace class, i.e.,* $\mathrm{Tr}\,(\mathrm{Op}(\mathbb{K}_O)) < \infty$*, since the kernel* $\mathbb{K}_O$ *is such that* $|\mathbb{K}_O(\eta,\eta;\omega_0)| \in L^1(\mathbb{R}_\eta)$*.* L^1*-integrability of the kernel* $\mathbb{K}_O$ *results from the fact that* $\mathfrak{J}_{0\mu}(\eta) = O(1/\sqrt{|\eta|})$ *as* $\eta \to \pm\infty$*, if* $1 + \delta - \sigma > 0$*. Therefore, the classical Fredholm theory*[37,63,84,19,62,65,64,66,97,94,46,48,95,83] *(the so-called 1st, 2nd, and 3rd theorems of Fredholm) applies to the operator* $\mathrm{Op}(\mathbb{K}_O)$*. Eigenvalues of the operator* $I - \mathrm{Op}(\mathbb{K}_O)$ *are the zeros of the Fredholm determinant defined as a series of determinant of infinite matrices (which can be rewritten in term of traces only of the operators* $\mathrm{Op}(\mathbb{K}_O)$ *and its higher iterates) while the eigenfunctions can be computed using higher Fredholm minors. If* $\delta = 0$ *(i.e., there exists a value of the parameter* μ *such that* $\mu = 0$ *or* $\mathfrak{J}_{0\mu} = 1$*: no gyroaveraging) and* $\sigma = 1$*, then the operator* $\mathrm{Op}(\mathbb{K}_O)$ *is no more in the trace class. Nevertheless, the Fredholm theory still holds, provided the classical Fredholm determinants are replaced by regularized ones. The latter are the so-called Hilbert-Carleman determinants of infinite matrices, setting to zero the main diagonal terms (where otherwise would appear the meaningless trace of the operator).*[63,19,62,94,95,31,47,48,50]

As a consequence of Proposition 8, we get the analytic Fredholm theorem for the open-contour operator in $\mathbb{C}^+$.

Theorem 3. *Let us suppose that assumptions of Lemma 1 are satisfied. Let* Ω *be any open connected subset of* $\mathbb{C}^+$*. Then either*

(i) $I - \mathrm{Op}(\mathbb{K}_O)$ *is nowhere invertible in* Ω*, or*
(ii) *the resolvent* $(I - \mathrm{Op}(\mathbb{K}_O))^{-1}$ *exists for all* $\omega_0 \in \Omega\backslash\mathscr{S}$*, where* $\mathscr{S}$ *is a discrete subset of* Ω *constituted of a countable number of isolated points (i.e., a set which has no limit points in* Ω*, and contains a finite number of singular points — poles — in each compact subset of* Ω*). In the latter case, the resolvent* $(I - \mathrm{Op}(\mathbb{K}_O))^{-1}$ *is meromorphic in* Ω*, analytic in* $\Omega\backslash\mathscr{S}$*, and the residues at the poles are finite rank operators (i.e., the invariant algebraic eigenspaces are finite dimensional). If* $\omega_0 \in \mathscr{S}$*, then the equations* $(I - \mathrm{Op}(\mathbb{K}_O)(\omega_0))\phi = 0$*, and* $(I - \mathrm{Op}(\mathbb{K}_O)^\star(\omega_0))\psi = 0$ *have the same number of linearly independent solutions; these are nonzero in* $L^2(\mathbb{R})$*, and hence almost everywhere. Moreover, the poles of* $(I - \mathrm{Op}(\mathbb{K}_O)(\omega_0, x))^{-1}$ *in the* ω_0*-complex plane depend continuously on* x *and can appear and disappear only at the boundary of* Ω*.*

Proof. From Proposition 8 we infer that $\mathrm{Op}(\mathbb{K}_O)(\omega_0) : \Omega \longmapsto \mathscr{L}(L^2)$ is an analytic operator-valued function, such that $\mathrm{Op}(\mathbb{K}_O)(\omega_0)$ is compact for each $\omega_0 \in \Omega$. This, together with the analytic Fredholm theorem such as Theorem VI.14 of Ref. 89 or Theorem 1 of Ref. 97 implies Theorem 3. The last assertion of Theorem 3 is a consequence of Theorem 3 of Ref. 96 and the fact that $\mathrm{Op}(\mathbb{K}_O)(\omega_0, x)$ is a family of compact operators jointly continuous in (ω_0, x) for each $(\omega_0, x) \in \Omega \times [x_{\min}, x_{\max}]$. □

We notice that Theorem 3 only asserts existence of eigenmodes for $\mathrm{Im}\,\omega_0 > 0$, which is the case that interests us most; indeed we aim at characterizing the plasma micro-instabilities driven by density and temperature gradients. Nevertheless, we would like to extend the kernel $\mathbb{K}$ of the integral equation (126) to eigenfrequencies of the non-positive imaginary part. The only obstacle for this comes from the divergent part of integral (126), associated to the kernel $\mathbb{K}_O$; indeed for $\mathrm{Im}\,\omega_0 \leq 0$ the exponential term in (127) is not integrable. In order to extend Theorem 3, we need to make an analytic continuation of the operator $\mathrm{Op}(\mathbb{K}_O)$, which is the purpose of the next proposition. We will find that the analytic continuation of the kernel $\mathbb{K}_O(\omega_0)$ with $\omega_0 \in \mathbb{C}^+$ to the half ω_0-complex plane $\mathbb{C}^-$ is $\mathbb{K}_O(\omega_0^\star)$, where $(\cdot)^\star$ denotes complex conjugation. If we consider only open contours, this means that the spectrum will be symmetric with respect to the real axis, and that the eigenmodes with eigenfrequency of negative imaginary part are the same as those with the eigenfrequency of positive imaginary part, but for negative time. Here, the analytic continuation allows us to recover time reversibility broken by the way boundary conditions were chosen. Indeed, in Sec. IV B, when integrating the perturbed Hamiltonian $h^{\pm}_{0\mu b\omega n}$ (see (73)) for open contours, we have two possible choices for the boundary conditions. The first choice, which has been done in Sec. IV B, consists of taking $\eta_0 = -\infty$ (resp. $\eta_0 = +\infty$) for $h^{+}_{0\mu b\omega n}$ (resp. $h^{-}_{0\mu b\omega n}$). Therefore, the kernel $\mathbb{K}_O$ is integrable only for $\omega_0 \in \mathbb{C}^+$. The second choice consists of taking $\eta_0 = +\infty$ (resp. $\eta_0 = -\infty$) for $h^{+}_{0\mu b\omega n}$ (resp. $h^{-}_{0\mu b\omega n}$). Therefore, the kernel $\mathbb{K}_O$ is integrable only for $\omega_0 \in \mathbb{C}^-$. As a consequence, damped eigenmodes, which must be seen as resonances or pseudo-eigenmodes, are described here by mixing of real frequencies (as a sum of all eigenmodes with real eigenfrequency ω_0). This is usual in waterbag descriptions[10] (where Landau damping is obtained as sums of plane waves with real frequencies) and is reminiscent of the Van Kampen-Case resolution of the eigenvalue problem.[105,20]

Before dealing with analytic continuation, let us look at the case where the frequency ω_0 is purely real. The kernel $\mathbb{K}_O$ is then no more integrable because of the loss of the exponential decay property. Nevertheless, using a well-suited change of unknowns, we can retrieve an integrable kernel. For this purpose, we introduce the following Hilbert space:

$$L^2_\kappa(\mathbb{R}) = \left\{ f : \eta \in \mathbb{R} \longmapsto f(\eta) \in \mathbb{R}, \ \text{s.t.} \ \|f\|_{L^2_\kappa(\mathbb{R})} = \langle f, f \rangle^{1/2}_{L^2_\kappa(\mathbb{R})} < \infty \right\},$$

where the scalar product $\langle \cdot, \cdot \rangle_{L^2_\kappa(\mathbb{R})}$ is defined by

$$\langle f, g \rangle_{L^2_\kappa(\mathbb{R})} = \int_{\mathbb{R}} f(\eta) g(\eta) \kappa^2(\eta) d\eta.$$

Here, the weight function $\eta \longmapsto \kappa(\eta)$ is given by

$$\kappa(\eta) = (1 + |\eta|^2)^\alpha, \quad \alpha \in \left(\frac{1-\delta}{4} + \frac{\sigma}{2}, \frac{3+\delta}{4} \right). \tag{136}$$

Making the change of unknowns $\phi_\kappa = \phi\kappa$, we can rewrite the integral equation (126) as

$$\phi_\kappa(\theta) = \int_{-\infty}^{\infty} d\eta \, \mathcal{K}(\theta, \eta; \omega_0) \phi_\kappa(\eta) = \mathrm{Op}(\mathcal{K}) \phi_\kappa(\theta),$$

where

$$\mathcal{K}(\theta, \eta; \omega_0) = \mathcal{K}_O(\theta, \eta; \omega_0) + \mathcal{K}_C(\theta, \eta; \omega_0) = \sum_{\mu b \in O} \mathcal{K}_{O\mu b}(\theta, \eta; \omega_0) + \sum_{\mu b \in C} \mathcal{K}_{C\mu b}(\theta, \eta; \omega_0)$$

and

$$\mathcal{K}_{O\mu b}(\theta, \eta; \omega_0) = \mathbb{K}_{O\mu b}(\theta, \eta; \omega_0) \frac{\kappa(\theta)}{\kappa(\eta)} \quad \text{and} \quad \mathcal{K}_{C\mu b}(\theta, \eta; \omega_0) = \mathbb{K}_{C\mu b}(\theta, \eta; \omega_0) \frac{\kappa(\theta)}{\kappa(\eta)}.$$

The extension to real frequency ω_0 relies on the following lemma.

Lemma 2. Proposition 8 and Theorem 3 can be extended to purely real frequencies ω_0. In other words, the operator-valued function $\mathrm{Op}(\mathbb{K}_O)(\omega_0) : \mathbb{C}^+ \cup \mathbb{R} \longmapsto \mathcal{L}(L^2_\kappa(\mathbb{R}))$ *(respectively,* $\mathrm{Op}(\mathcal{K}_O)(\omega_0) : \mathbb{C}^+ \cup \mathbb{R} \longmapsto \mathcal{L}(L^2(\mathbb{R}))$*) constitutes a Hilbert-Schmidt family, hence a family of compact operators on the Hilbert space $L^2_\kappa(\mathbb{R})$ (resp. $L^2(\mathbb{R})$).*

Proof. Using the previous new formulation and estimate (135), we observe that the kernel $\mathcal{K}_O \in L^2(\mathbb{R}_\eta \times \mathbb{R}_\theta)$ for $\omega_0 \in \mathbb{C}^+ \cup \mathbb{R}$. The properties of the kernel $\mathcal{K}_C$ will be studied later. Using the change of variables $\tilde{\eta} = \eta - \theta$ in (135), and the asymptotic property $\mathfrak{J}_{0\mu}(\eta) = O(1/\sqrt{|\eta|})$ as $\eta \to \pm\infty$, we find that there exist two constants $C_1 = C_1(\mu, b, \omega_0)$ and $C_2 = C_2(\mu, b)$ such that

$$|\mathcal{K}_{O\mu b}(\theta, \eta; \omega_0)|^2 \leq C_1(1 + |\theta|^2)^{-2-\delta/2+2\alpha}(1 + |\tilde{\eta} + \theta|^2)^{\sigma-\delta/2-2\alpha} \exp\left(-C_2 \operatorname{Im}\omega_0 |\tilde{\eta}|\right), \quad (137)$$

and thus $|\mathcal{K}_O(\theta, \eta; \omega_0)|^2 \in L^1(\mathbb{R}_\eta \times \mathbb{R}_\theta)$ for $\omega_0 \in \mathbb{C}^+ \cup \mathbb{R}$, as long as $(1-\delta)/4 + \sigma/2 < \alpha < (3+\delta)/4$. □

Analytic continuation to the lower half-complex plane relies on the following proposition.

Proposition 9. Let us set $\mathbb{C}^- = \mathbb{C}\backslash\{\omega_0 \in \mathbb{C} \,|\, \operatorname{Im}\omega_0 \geq 0\}$ *and let* $\omega_0^\star$ *be the complex conjugate of* ω_0*. Then, for every* $\omega_0 \in \mathbb{C}^-$*, the operator* $\operatorname{Op}(\mathcal{K}_O)(\omega_0^\star)$ *(resp.* $\operatorname{Op}(\mathbb{K}_O)(\omega_0^\star)$*) constitutes an analytic continuation of the operator* $\operatorname{Op}(\mathcal{K}_O)(\tilde{\omega}_0)$ *(resp.* $\operatorname{Op}(\mathbb{K}_O)(\tilde{\omega}_0)$*) for* $\tilde{\omega}_0 \in \mathbb{C}^+$*. Moreover* $\operatorname{Op}(\mathcal{K}_O)(\omega_0^\star) : \mathbb{C}^- \longmapsto \mathscr{L}(L^2)$ *(resp.* $\operatorname{Op}(\mathbb{K}_O)(\omega_0^\star) : \mathbb{C}^- \longmapsto \mathscr{L}(L^2_\kappa)$*) is an analytic operator-valued function such that* $\operatorname{Op}(\mathcal{K}_O)(\omega_0^\star)$ *(resp.* $\operatorname{Op}(\mathbb{K}_O)(\omega_0^\star)$*) is a Hilbert-Schmidt operator, hence compact on* $L^2(\mathbb{R})$ *(resp.* $L^2_\kappa(\mathbb{R})$*) for each* $\omega_0 \in \mathbb{C}^-$*.*

Proof. To show that the operator $\operatorname{Op}(\mathcal{K}_O)(\omega_0^\star)$ with $\omega_0 \in \mathbb{C}^-$ is an analytic continuation of the operator $\operatorname{Op}(\mathcal{K}_O)(\omega_0)$ with $\omega_0 \in \mathbb{C}^+$, we write the integral equation (126) (restricted to the kernel $\mathcal{K}_O$) in new variables, with new unknowns. Let us define, for all $(\mu, b) \in O$,

$$T_{O\mu b} = \int_{-\pi}^{\pi} d\eta \left(\frac{qR}{a_{\mu b} b_\varphi}\right)(\eta), \quad \text{and} \quad \omega_{O\mu b} = \frac{2\pi}{T_{O\mu b}}.$$

Since $\{a_{\mu b} > 0\}_{(\mu,b)\in O}$, we define the monotone and invertible change of variables

$$\eta \longleftrightarrow \theta_{\mu b}(\eta), \quad \theta_{\mu b}(\eta) = \omega_{O\mu b} \int_0^\eta d\eta' \left(\frac{qR}{a_{\mu b} b_\varphi}\right)(\eta').$$

Let us then define for all $(\mu, b) \in O$,

$$\Omega_{d\mu b} = -\int_0^\eta d\eta' \left(\frac{qR}{a_{\mu b} b_\varphi}\right)(\eta')\,\omega_{d\mu b}(\eta'), \quad \text{and} \quad \Omega_{\diamond\mu b} = -\int_0^\eta d\eta' \left(\frac{qR}{a_{\mu b} b_\varphi}\right)(\eta')\,\omega_{\diamond\mu b}(\eta').$$

Now we introduce the unknowns $\psi_{\mu b\pm}(\theta_{\mu b})$ defined as

$$\begin{aligned}\psi_{\mu b\pm}(\theta_{\mu b}) &= \Psi_{\mu b\pm}(\eta(\theta_{\mu b})) \\ &:= \exp\left(\mp \mathrm{i}\left[\Omega_{d\mu b}(\eta(\theta_{\mu b})) - \mathrm{i}\Omega_{\diamond\mu b}(\eta(\theta_{\mu b}))\right]\right)\left(\frac{e}{T_{e0}}\Omega_{*\mu b}(\omega_0) a_{\mu b}^2 \frac{\mathfrak{J}_{0\mu}}{\kappa}\phi_\kappa\right)(\eta(\theta_{\mu b})). \quad (138)\end{aligned}$$

Let us suppose that $\psi_{\mu b\pm} \in L^2(\mathbb{R}_{\theta_{\mu b}})$. Since $\mathfrak{J}_{0\mu}(\eta) = O(1/\sqrt{|\eta|})$ as $\eta \to \pm\infty$, and $a_{\mu b}^2 \Omega_{*\mu b}(\eta) \lesssim (1 + |\eta|^2)^{\sigma/2}$ (see Remark 20), this is the case if either $(1 + |\eta|^2)^{\sigma/2-\delta/4}\phi \in L^2(\mathbb{R}_\eta)$, or the Fourier transform $\widehat{\phi} \in H^{\sigma-\delta/2}(\mathbb{R})$. Therefore we can introduce the Fourier transform $\widehat{\psi}_{\mu b\pm}(\lambda)$ of $\psi_{\mu b\pm}(\theta_{\mu b})$ defined by

$$\widehat{\psi}_{\mu b\pm}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\theta_{\mu b}\, \exp(-\mathrm{i}\lambda\theta_{\mu b})\psi_{\mu b\pm}(\theta_{\mu b}). \quad (139)$$

Roughly speaking, the analytical continuation away from the real axis of the functions $\widehat{\psi}(\lambda)$ depends on the decreasing and regularity properties of the function $\phi(\eta)$. If $\phi(\eta)$ is square summable and exponentially decreasing at infinity then it is the same for $\psi(\theta)$ and thus its Fourier transform $\widehat{\psi}(\lambda)$ belongs to the Hardy space H^2. We recall that the Hardy space H^2 consists of functions which are analytic in a strip of the complex plane containing the real axis, and which are square summable on any line parallel to the real axis within this strip (see Theorem I of Ref. 80). Finally, we also have Paley-Wiener type results that for a tempered distribution $\widehat{\psi}$ on $\mathbb{R}$ to be the Fourier transform of a compactly supported distribution (resp. $\mathscr{C}^\infty$ function) ψ, it is necessary and sufficient for $\widehat{\psi}$ to be a $\mathscr{C}^\infty$ function slowly growing at infinity (resp. rapidly decaying at infinity, i.e., belonging to $\mathcal{S}$) extendable to $\mathbb{C}$ as an entire function of exponential type (i.e., with some exponential growing properties as the imaginary part tends to infinity). We refer to the Paley-Wiener (-Schwartz) theorems in Refs. 42 and 41 for precise results. Using definitions (138) and (139), equations (65) become

$$h^{\pm}_{0\mu b\omega n}(\eta) = \mp \mathrm{i} \exp\left(\mp \mathrm{i}\left[-\Omega_{d\mu b}(\eta) + \mathrm{i}\Omega_{\diamond\mu b}(\eta)\right]\right) \frac{1}{\omega_{O\mu b}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda\, \widehat{\psi}_{\mu b\pm}(\lambda) \exp(\mathrm{i}\lambda\theta_{\mu b}(\eta)) \int_{\mp\infty}^{0} d\widetilde{\eta} \exp\left(\mathrm{i}\left(\lambda \mp \frac{\omega_0}{\omega_{O\mu b}}\right)\widetilde{\eta}\right).$$

Since $\mathrm{Im}\,\omega_0 > 0$, we can perform the $\widetilde{\eta}$-integration in the latter equation and we obtain

$$\phi^L_\kappa(\eta) = \left(\mathrm{Op}(\mathcal{K}_O)(\omega_0)\phi_\kappa\right)(\eta) = -\sum_{\mu b \in O} 2\pi \frac{\Omega_i}{q_i} \frac{\mathcal{A}_{\mu b}}{\omega_{O\mu b}} \frac{\kappa(\eta)}{Q(\eta)} \frac{\mathfrak{I}_{0\mu}(\eta)}{a_{\mu b}(\eta)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda \exp(\mathrm{i}\lambda\theta_{\mu b}(\eta)) \left\{ \frac{\exp\left(\mathrm{i}\left[\Omega_{d\mu b}(\eta) - \mathrm{i}\Omega_{\diamond\mu b}(\eta)\right]\right)}{\lambda - \omega_0/\omega_{O\mu b}} \widehat{\psi}_{\mu b+}(\lambda) - \frac{\exp\left(-\mathrm{i}\left[\Omega_{d\mu b}(\eta) - \mathrm{i}\Omega_{\diamond\mu b}(\eta)\right]\right)}{\lambda + \omega_0/\omega_{O\mu b}} \widehat{\psi}_{\mu b-}(\lambda) \right\}. \quad (140)$$

From definition (138) we observe that $\psi_{\mu b\pm}$ depends on ω_0, i.e., $\psi_{\mu b\pm} = \psi_{\mu b\pm}(\theta_{\mu b}; \omega_0 = \omega_0^{\mathfrak{R}} + \mathrm{i}\omega_0^{\mathfrak{I}})$. Let us set $\psi_{\star\mu b\pm} = \psi_{\mu b\pm}(\theta_{\mu b}; \omega_0^\star)$ (resp. $\widehat{\psi}_{\star\mu b\pm}$ using (139)) and $\psi_{\mathfrak{R}\mu b\pm} = \psi_{\mu b\pm}(\theta_{\mu b}; \omega_0^{\mathfrak{R}})$ (resp. $\widehat{\psi}_{\mathfrak{R}\mu b\pm}$ using (139)). Then, for $\mathrm{Im}\,\omega_0 < 0$, we define ϕ^R_κ as

$$\phi^R_\kappa(\eta) = \left(\mathrm{Op}(\mathcal{K}_O)(\omega_0^\star)\phi_\kappa\right)(\eta) = -\sum_{\mu b \in O} 2\pi \frac{\Omega_i}{q_i} \frac{\mathcal{A}_{\mu b}}{\omega_{O\mu b}} \frac{\kappa(\eta)}{Q(\eta)} \frac{\mathfrak{I}_{0\mu}(\eta)}{a_{\mu b}(\eta)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda \exp(\mathrm{i}\lambda\theta_{\mu b}(\eta)) \left\{ \frac{\exp\left(\mathrm{i}\left[\Omega_{d\mu b}(\eta) - \mathrm{i}\Omega_{\diamond\mu b}(\eta)\right]\right)}{\lambda - \omega_0^\star/\omega_{O\mu b}} \widehat{\psi}_{\star\mu b+}(\lambda) - \frac{\exp\left(-\mathrm{i}\left[\Omega_{d\mu b}(\eta) - \mathrm{i}\Omega_{\diamond\mu b}(\eta)\right]\right)}{\lambda + \omega_0^\star/\omega_{O\mu b}} \widehat{\psi}_{\star\mu b-}(\lambda) \right\}. \quad (141)$$

From (138) and (140) and (141), we obtain that (140) is analytic with respect to ω_0 in $\mathbb{C}^+$ and (141) is analytic with respect to ω_0 in $\mathbb{C}^-$. Now using the Sokhotski-Plemelj formula,[76] i.e.,

$$\frac{1}{\lambda - \omega_0^{\mathfrak{R}}/\omega_{O\mu b} \pm \mathrm{i}0^+} = \mathrm{p.v.}\left(\frac{1}{\lambda - \omega_0^{\mathfrak{R}}/\omega_{O\mu b}}\right) \mp \mathrm{i}\pi\delta\left(\lambda - \omega_0^{\mathfrak{R}}/\omega_{O\mu b}\right),$$

we obtain that the boundary values of ϕ^L_κ as $\mathrm{Im}\,\omega_0 \to 0^+$ and ϕ^R_κ as $\mathrm{Im}\,\omega_0 \to 0^-$ are equal, i.e., $\phi^L_\kappa(\omega_0^{\mathfrak{R}}) = \phi^R_\kappa(\omega_0^{\mathfrak{R}})$. Therefore, from the principle of analytic continuation (see, for example, Ref. 79), ϕ^R_κ constitutes the unique analytic continuation in $\mathbb{C}^-$ of the function ϕ^L_κ analytic in $\mathbb{C}^+$. Let us note that the boundary values of ϕ^L_κ (as $\mathrm{Im}\,\omega_0 \to 0^+$) or ϕ^R_κ (as $\mathrm{Im}\,\omega_0 \to 0^-$) involve the Hilbert integral (with singular Cauchy kernel)

$$\mathrm{p.v.}\int_{\mathbb{R}} \frac{f(y)}{y - x} dy,$$

which defines a bounded operator from L^p into itself, with $1 < p < \infty$ (see for instance Part II, Chapter XI, Section 7 of Ref. 31, or Chapter V of Ref. 102). Therefore, the boundary values of ϕ^L_κ and ϕ^R_κ on the real axis of the ω_0-complex plane are well defined and equal in $L^2(\mathbb{R})$ as long as $(1 + |\eta|^2)^{\sigma/2-\delta/4}\phi \in L^2(\mathbb{R})$. Assuming $(1 + |\eta|^2)^{1/2(\sigma-\delta/2+1/2+\varepsilon)}\phi \in L^2(\mathbb{R}_\eta)$, with $\varepsilon > 0$, or equivalently the Fourier transform $\widehat{\phi} \in H^{\sigma-\delta/2+1/2+\varepsilon}(\mathbb{R})$, we obtain that the boundary values of ϕ^L_κ and ϕ^R_κ on the real axis of the ω_0-complex plane are equal in $H^{1/2+\varepsilon}(\mathbb{R})$. Therefore, using continuous Sobolev embeddings theorem, the boundary values of ϕ^L_κ and ϕ^R_κ on the real axis are equal in the space of continuous functions. Finally, assuming that ϕ_κ is square summable and rapidly decreasing (or compactly supported), we find that boundary values of ϕ^L_κ and ϕ^R_κ on the real axis of the ω_0-complex plane are well defined and equal in the space of infinitely differentiable functions. Moreover, following the proof of Proposition 8, we obtain that $\mathrm{Op}(\mathcal{K}_O)(\omega_0^\star) : \mathbb{C}^- \longmapsto \mathcal{L}(L^2)$ (resp. $\mathrm{Op}(\mathbb{K}_O)(\omega_0^\star) : \mathbb{C}^- \longmapsto \mathcal{L}(L^2_\kappa)$) is an analytic operator-valued function such that $\mathrm{Op}(\mathcal{K}_O)(\omega_0^\star)$ (respectively, $\mathrm{Op}(\mathbb{K}_O)(\omega_0^\star)$) is a Hilbert-Schmidt operator; it is thus compact on $L^2(\mathbb{R})$ (respectively, $L^2_\kappa(\mathbb{R})$) for each $\omega_0 \in \mathbb{C}^-$; which ends the proof. □

We can now state the analytic Fredholm theorem for the open-contour operator in $\mathbb{C}$.

Theorem 4. *Let us suppose that assumptions of Lemma 1 are satisfied. Let* Ω *be any open connected subset of* $\mathbb{C}$*, and* $\mathrm{Op}(\mathcal{K}_O)$ *be the analytic extended operator obtained in Proposition 9. Then either*

(i) $I - \mathrm{Op}(\mathcal{K}_O)$ *is nowhere invertible in* Ω*, or*
(ii) *the resolvent* $(I - \mathrm{Op}(\mathcal{K}_O))^{-1}$ *exists for all* $\omega_0 \in \Omega\backslash\mathscr{S}$*, where* $\mathscr{S}$ *is a discrete subset of* Ω *constituted of a countable number of isolated points. In the latter case the resolvent* $(I - \mathrm{Op}(\mathcal{K}_O))^{-1}$ *extends to an operator-valued function in* ω_0*, which is analytic in* $\Omega\backslash\mathscr{S}$*, meromorphic in* Ω*, and such that the residues at the poles are finite rank operators. If* $\omega_0 \in \mathscr{S}$*, then the equations* $(I - \mathrm{Op}(\mathcal{K}_O)(\omega_0))\phi = 0$ *and* $(I - \mathrm{Op}(\mathcal{K}_O)^\star(\omega_0))\psi = 0$ *have the same number of linearly independent solutions; these are non-zero in* $L^2(\mathbb{R})$ *and hence almost everywhere. Moreover the poles of* $(I - \mathrm{Op}(\mathcal{K}_O)(\omega_0, x))^{-1}$ *in the* ω_0*-complex plane, depend continuously on* x *and can appear and disappear only at the boundary of* Ω*.*

Proof. From Propositions 8 and 9 we find that $\mathrm{Op}(\mathcal{K}_O)(\omega_0) : \Omega \longmapsto \mathscr{L}(L^2)$ is an analytic operator-valued function such that $\mathrm{Op}(\mathcal{K}_O)(\omega_0)$ is compact for each $\omega_0 \in \Omega$. This, together with the analytic Fredholm theorem such as Theorem VI.14 of Ref. 89 or Theorem 1 of Ref. 97, implies Theorem 4. The last assertion of Theorem 4 is a consequence of Theorem 3 of Ref. 96 and the fact that $\mathrm{Op}(\mathcal{K})(\omega_0, x)$ is a family of compact operators jointly continuous in (ω_0, x) for each $(\omega_0, x) \in \Omega \times [x_{\min}, x_{\max}]$. □

Remark 22. A theorem similar to Theorem 4 can be stated for the analytical continuation of the operator-valued function $I - \mathrm{Op}(\mathbb{K}_O)(\omega_0) : \Omega \longmapsto \mathscr{L}(L^2_\kappa(\mathbb{R}))$*, in the Hilbert space* $L^2_\kappa(\mathbb{R})$*.*

Remark 23. Let us note that the transformation, introduced in the proof of Proposition 9 can be used to transform the eigenvalue problem with nonlinear eigenparameter ω_0 *dependence, into an eigenvalue problem (for instance, a generalized eigenvalue problem of higher dimension) with linear eigenparameter* ω_0 *dependence. This strategy seems to work only for open contours. We can also use similar Fourier transforms for closed contours, but the nonlinear trigonometric terms* $\sin \mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b}, \theta^{+,\ell}_{L\mu b}; \omega_0)$ *in* $\mathbb{K}_C$ *(128) cannot really disappear.*

Let us recover the generalized eigenvalue problem (linear in ω_0*) for open contours, i.e., for all* $(\mu, b) \in O$*. We consider only the case for which* $\omega_0 \in \mathbb{C}^+$*, but obviously we can recover this reformulation straightforwardly for* $\omega_0 \in \mathbb{C}^-$*. Taking (140), multiplying it by*

$$\frac{\omega_{O\mu'b'}}{\sqrt{2\pi}} \exp\left(\mp \mathrm{i}\left[\Omega_{d\mu'b'}(\eta) - \mathrm{i}\Omega_{\diamond\mu'b'}(\eta)\right] \exp\left(-\mathrm{i}\xi\theta_{\mu'b'}(\eta)\right)\right) \left(\frac{e}{T_{e0}} \frac{qR}{b_\varphi} \Omega_{*\mu'b'}(\omega_0) a^2_{\mu'b'} \frac{\mathfrak{J}_{0\mu'}}{\kappa}\right)(\eta),$$

and integrating the result with respect to η*, we obtain*

$$\begin{aligned}\widehat{\psi}_{\mu'b'\pm}(\xi) = -\int_{-\infty}^{\infty} d\lambda \sum_{\mu b \in O} \int_{\mathbb{R}} d\eta\, \frac{\Omega_i}{q_i} \frac{\mathcal{A}_{\mu b}}{Q(\eta)} \frac{\mathfrak{J}_{0\mu}(\eta)}{a_{\mu b}(\eta)} \frac{\omega_{O\mu'b'}}{\omega_{O\mu b}} \exp\left(\mp \mathrm{i}\left[\Omega_{d\mu'b'}(\eta) - \mathrm{i}\Omega_{\diamond\mu'b'}(\eta)\right]\right) \\ \left(\frac{e}{T_{e0}} \frac{qR}{b_\varphi} \Omega_{*\mu'b'}(\omega_0) a^2_{\mu'b'} \mathfrak{J}_{0\mu'}\right)(\eta) \exp(\mathrm{i}[\lambda\theta_{\mu b}(\eta) - \xi\theta_{\mu'b'}(\eta)]) \\ \left\{\frac{\exp\left(\mathrm{i}\left[\Omega_{d\mu b}(\eta) - \mathrm{i}\Omega_{\diamond\mu b}(\eta)\right]\right)}{\lambda - \omega_0/\omega_{O\mu b}} \widehat{\psi}_{\mu b+}(\lambda) - \frac{\exp\left(-\mathrm{i}\left[\Omega_{d\mu b}(\eta) - \mathrm{i}\Omega_{\diamond\mu b}(\eta)\right]\right)}{\lambda + \omega_0/\omega_{O\mu b}} \widehat{\psi}_{\mu b-}(\lambda)\right\}. \quad (142)\end{aligned}$$

Let us define the set $\widehat{O} = O \times \{+, -\}$ *and the index* $\ell = (\mu, b, \alpha)$*, where* $(\mu, b) \in O$ *and* $\alpha \in \{+, -\}$*. Let us introduce the unknowns* $\Psi_{\mu b\pm}(\lambda) = \widehat{\psi}_{\mu b\pm}(\lambda)/(\lambda \mp \omega_0/\omega_{O\mu b})$*, and the unknown vector* $\Psi = (\Psi_\ell)^T_{\ell \in \widehat{O}}$*. Finally, we define the diagonal matrix* $\Omega_{\widehat{O}} = \mathrm{Diag}(\{-\alpha\omega^{-1}_{O\mu b}\}_{(\mu,b,\alpha)\in\widehat{O}})$*. Since in (142) the term* $\Omega_{*\mu'b'}(\omega_0)$ *is linear in* ω_0*, we can recast the eigenvalue problem (126) with nonlinear eigenparameter dependence as the following generalized eigenvalue problem, with linear dependence in the eigenparameter* ω_0*:*

$$\mathcal{A}\Psi - \omega_0 \mathcal{B}\Psi = 0.$$

Here, $\mathcal{A}$ and $\mathcal{B}$ are linear operators written as follows:

$$(\mathcal{A}\Psi)(\xi) = \xi I_{\widehat{O}}\Psi(\xi) - \int_{\mathbb{R}} d\lambda\, \mathcal{K}_{\mathcal{A}}(\xi,\lambda)\Psi(\lambda), \quad (\mathcal{B}\Psi)(\xi) = \Omega_{\widehat{O}}\Psi(\xi) - \int_{\mathbb{R}} d\lambda\, \mathcal{K}_{\mathcal{B}}(\xi,\lambda)\Psi(\lambda),$$

where the matrix-valued kernels $\mathcal{K}_{\mathcal{A}}$ and $\mathcal{K}_{\mathcal{B}}$ are independent of ω_0 and whose exact expressions can be easily inferred from (142).

3. Case of open and closed contours

In this section we consider both open and closed contours. The main results of this section are Theorem 5 (with hypothesis (**H**) and the spectrum in $\mathbb{C}\backslash\Sigma$, where Σ is a subset of the real axis defined by (143)), Theorem 6 (without hypothesis (**H**) and the spectrum in $\mathbb{C}^+$) and Theorem 7 (without hypothesis (**H**) and the spectrum in $\mathbb{C}\backslash\Sigma$), which are established using Propositions 10 and 11.

Let us deal with the operator $\mathrm{Op}(\mathbb{K}_C)$, associated to closed contours. We remark that for closed contours the parameter μ is positive. From (128), we observe that the kernel $\mathbb{K}_C$ is singular in ω_0 when $\sin\mathcal{I}_{\mu b}(-\theta_{L\mu b},\theta_{L\mu b};\omega_0) = \sin\mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b};\omega_0)$ for all $\ell\in\mathbb{Z}$, vanishes. By (131), this happens whenever

$$\omega_0 = \bar{\omega}_{Cd\mu b} + \left(l+\frac{1}{2}\right)\omega_{C\mu b}, \quad \forall l\in\mathbb{Z}, \quad (\mu,b)\in C.$$

Here we have set, for all $(\mu,b)\in C$,

$$T_{C\mu b} = \int_{-\theta_{L\mu b}}^{\theta_{L\mu b}} d\eta \left(\frac{qR}{a_{\mu b}b_\varphi}\right)(\eta), \quad \omega_{C\mu b} = \frac{2\pi}{T_{C\mu b}},$$
$$\bar{\omega}_{Cd\mu b} = \frac{1}{T_{C\mu b}}\int_{-\theta_{L\mu b}}^{\theta_{L\mu b}} d\eta\, \omega_{d\mu b}(\eta)\left(\frac{qR}{a_{\mu b}b_\varphi}\right)(\eta).$$

We define the set Σ by

$$\Sigma = \left\{\omega_0\in\mathbb{R}\,|\,\omega_0 = \bar{\omega}_{Cd\mu b} + \left(l+\frac{1}{2}\right)\omega_{C\mu b}, \quad \forall l\in\mathbb{Z}, \quad (\mu,b)\in C\right\}. \tag{143}$$

We note that this set Σ depends on the radial variable x, i.e., $\Sigma = \Sigma(x)$. Since the set Σ contains the poles of the kernel $\mathbb{K}_C$, the operator $\mathrm{Op}(\mathbb{K}_C)(\omega_0) : \mathbb{C} \longmapsto \mathcal{L}(L^2)$ is a meromorphic operator-valued function of $\mathbb{C}$. Let us fix $\omega_0\in\mathbb{C}$, say a regular point of analyticity of the meromorphic operator-valued function $\mathrm{Op}(\mathbb{K}_C)(\omega_0) : \mathbb{C} \longmapsto \mathcal{L}(L^2)$. When we look at the integral operator $I - \mathrm{Op}(\mathbb{K}_C)$ (see (128)) we observe that roughly speaking the integral operator is weakly singular with an algebraic singularity of power minus one-half due to the equilibrium closed contours (see (134)). This suggests that the operator should be compact on some well-suited Hilbert spaces (see Ref. 71 or section 9.5 of Ref. 33). In addition, the semi-separable structure of the kernel suggests that the operator is a Hilbert-Schmidt operator in some relevant Hilbert spaces (see Chapter IX[49]). In order to prove such properties, let us introduce the Hilbert space

$$L^2_\varrho(\mathbb{R}) = \left\{f : \eta\in\mathbb{R} \longmapsto f(\eta)\in\mathbb{R}, \text{ s.t. } \|f\|_{L^2_\varrho(\mathbb{R})} = \langle f,f\rangle^{1/2}_{L^2_\varrho(\mathbb{R})} < \infty\right\},$$

where the scalar product $\langle\cdot,\cdot\rangle_{L^2_\varrho(\mathbb{R})}$ is defined by

$$\langle f,g\rangle_{L^2_\varrho(\mathbb{R})} = \int_{\mathbb{R}} f(\eta)g(\eta)\varrho^2(\eta)d\eta.$$

Here, the weight function ϱ is given by

$$\varrho = \kappa\varpi, \quad \text{with} \quad \varpi = \left(\sum_{\mu b\in\mathscr{C}} \mathcal{A}_{\mu b}\frac{\mathfrak{I}^2_{0\mu}}{a_{\mu b}}\right)^\beta \in L^\gamma_{\mathrm{loc}}(\mathbb{R}), \ \beta\in(0,1), \ \gamma\in(0,2/\beta),$$

and κ is given by (136). Making the change of unknowns $\phi_\varrho = \phi\varrho$, we can recast the integral equation (126) as

$$\phi_\varrho(\theta) = \int_{-\infty}^{\infty} d\eta\, \mathcal{G}(\theta,\eta;\omega_0)\phi_\varrho(\eta) = \mathrm{Op}(\mathcal{G})\phi_\varrho(\theta),$$

where

$$\mathcal{G}(\theta,\eta;\omega_0) = \mathcal{G}_O(\theta,\eta;\omega_0) + \mathcal{G}_C(\theta,\eta;\omega_0) = \sum_{\mu b\in O} \mathcal{G}_{O\mu b}(\theta,\eta;\omega_0) + \sum_{\mu b\in C} \mathcal{G}_{C\mu b}(\theta,\eta;\omega_0),$$

and

$$\mathcal{G}_{O\mu b}(\theta,\eta;\omega_0) = \mathbb{K}_{O\mu b}(\theta,\eta;\omega_0)\frac{\varrho(\theta)}{\varrho(\eta)}, \quad \text{and} \quad \mathcal{G}_{C\mu b}(\theta,\eta;\omega_0) = \mathbb{K}_{C\mu b}(\theta,\eta;\omega_0)\frac{\varrho(\theta)}{\varrho(\eta)}. \tag{144}$$

If for every μ such that $(\mu,b)\in C$, the zeros of the map $\theta \longmapsto \mathfrak{I}_{0\mu}(\theta)$ are different from the limit angles $\{\theta^{\pm,\ell}_{L\mu b}\}_{\mu b\in C,\, \ell\in\mathbb{Z}}$, we can show that $\mathrm{Op}(\mathcal{G})(\omega_0) : \mathbb{C} \longmapsto \mathcal{L}(L^2)$ is a meromorphic operator-valued function of $\mathbb{C}$ where the coefficients of the Laurent series of $\mathrm{Op}(\mathcal{G})(\omega_0)$ are trace class and Hilbert-Schmidt, hence compact operators on $L^2(\mathbb{R})$. This feature relies on a specific behaviour of the denominator Q, which remains unexploited until now. This nice property fails if for every μ such that $(\mu,b)\in C$, the zeros of the map $\theta \longmapsto \mathfrak{I}_{0\mu}(\theta)$ coincide with the limit angles $\{\theta^{\pm,\ell}_{L\mu b}\}_{\mu b\in C,\, \ell\in\mathbb{Z}}$, because the zeros of $\mathfrak{I}^2_{0\mu}$ are of order larger than one-half and thus cancel those of $a_{\mu b}$. Therefore we make the following hypothesis.

Assumption H. For every μ such that $(\mu,b)\in C$, the roots of the map $\theta \longmapsto \mathfrak{I}_{0\mu}(\theta)$, do not belong to the set $\{\theta^{\pm,\ell}_{L\mu b}\}_{\mu b\in C,\, \ell\in\mathbb{Z}}$.

Under this assumption we obtain the following proposition.

*Proposition 10. Let us assume hypothesis (**H**) and assumptions of Lemma 1. Then* $\mathrm{Op}(\mathcal{G})(\omega_0)$: $\mathbb{C}\backslash\Sigma \longmapsto \mathcal{L}(L^2)$ *(resp.* $\mathrm{Op}(\mathbb{K})(\omega_0) : \mathbb{C}\backslash\Sigma \longmapsto \mathcal{L}(L^2_\varrho)$*) is an analytic operator-valued function such that* $\mathrm{Op}(\mathcal{G})(\omega_0)$ *(resp.* $\mathrm{Op}(\mathbb{K})(\omega_0)$*) is Hilbert-Schmidt, hence compact on* $L^2(\mathbb{R})$ *(resp.* $L^2_\varrho(\mathbb{R})$*) for each* $\omega_0\in\mathbb{C}\backslash\Sigma$*. Moreover,* $\mathrm{Op}(\mathcal{G})(\omega_0) : \mathbb{C} \longmapsto \mathcal{L}(L^2)$ *(resp.* $\mathrm{Op}(\mathbb{K})(\omega_0) : \mathbb{C} \longmapsto \mathcal{L}(L^2_\varrho)$*) is a meromorphic operator-valued function of* $\mathbb{C}$ *where the coefficients of the Laurent series of* $\mathrm{Op}(\mathcal{G})(\omega_0)$ *(resp.* $\mathrm{Op}(\mathbb{K})(\omega_0)$*) — in particular the residues at the simple poles* Σ *— are Hilbert-Schmidt, hence compact operators on* $L^2(\mathbb{R})$ *(resp.* $L^2_\varrho(\mathbb{R})$*).*

Proof. Let us first show that the operator $\mathrm{Op}(\mathcal{G}_O)$ belongs to the class of Hilbert-Schmidt operators with bounded trace for all values of ω_0 in the upper half-complex plane $\mathbb{C}^+$ and the real axis, i.e., that the kernel $\mathcal{G}_O \in L^2(\mathbb{R}\times\mathbb{R})$. Of course, we can extend it to the lower half-complex plane $\mathbb{C}^-$ by analytical continuation as has been done in Proposition 9. This requires that ϕ decay rapidly enough at infinity. If we replace $\mathcal{G}_{O\mu b}$ by $\mathcal{K}_{O\mu b}$, then estimate (137) for $\mathrm{Im}\,\omega_0 \geq 0$, still holds for all η, $\theta \in \mathcal{V}(\pm\infty)$ (the notation $\mathcal{V}(x)$ denotes a neighborhood of x). As a consequence, the kernel $|\mathcal{G}_{O\mu b}(\theta,\eta;\omega_0)|^2$ is integrable in the neighborhood of infinity as long as $(1-\delta/4)+\sigma/2 < \alpha < (3+\delta)/4$, and thus we only need to check that for all $(\mu,b)\in O$, $\mathcal{G}_{O\mu b} \in L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R})$ in the neighborhood of the limit angles $\{\theta^{\pm,\ell}_{L\mu b}\}_{\mu b\in C,\, \ell\in\mathbb{Z}}$. Under assumption (**H**), we obtain, for all $\eta\in\mathcal{V}(\theta^{s',\ell'}_{L\mu' b'})$ and $\theta\in\mathcal{V}(\theta^{s'',\ell''}_{L\mu'' b''})$,

$$|\mathcal{G}_{O\mu b}(\theta,\eta)|^2 \lesssim \frac{1}{\varpi(\eta)^2}\frac{1}{\varpi(\theta)^{2/\beta-2}} \lesssim \frac{a^{\beta}_{\mu' b'}(\eta)}{\mathfrak{I}^{2\beta}_{0\mu'}(\eta)}\frac{a^{1-\beta}_{\mu'' b''}(\theta)}{\mathfrak{I}^{2(1-\beta)}_{0\mu''}(\theta)} \in L^1\left(\mathcal{V}(\theta^{s',\ell'}_{L\mu' b'})\times\mathcal{V}(\theta^{s'',\ell''}_{L\mu'' b''})\right)$$

and thus the kernel $\mathcal{G}_{O\mu b}\in L^2(\mathbb{R}\times\mathbb{R})$, for all $(\mu,b)\in O$. Let us note that the same kind of estimates holds if some pairs of limit angles have identical values, since the corresponding equilibrium closed contours have the same algebraic singularity of power minus one-half. By (135), it is also clear that $\mathcal{G}_{O\mu b}(\eta,\eta)\in L^1(\mathbb{R}\times\mathbb{R})$ as $1+\delta-\sigma>0$, since the rescaling factor $\varrho(\theta)/\varrho(\eta)$ simplifies. Moreover from (144), analyticity of $\mathrm{Op}(\mathcal{G}_O)(\omega_0)$ (resp. $\mathrm{Op}(\mathbb{K}_O)(\omega_0)$) with respect to ω_0 is obvious since functionals of ω_0 involve only products of polynomials and exponential of ω_0. Therefore,

$\mathrm{Op}(\mathcal{G}_O)(\omega_0) : \mathcal{D} \longmapsto \mathscr{L}(L^2)$ (resp. $\mathrm{Op}(\mathbb{K}_O)(\omega_0) : \mathcal{D} \longmapsto \mathscr{L}(L^2_\varrho)$) forms an analytic operator-valued function on any open connected subset $\mathcal{D}$ of $\mathbb{C}$, such that the families of operators $\mathrm{Op}(\mathcal{G}_O)(\omega_0)$ (resp. $\mathrm{Op}(\mathbb{K}_O)(\omega_0)$) belong to the class of Hilbert-Schmidt operators in $L^2(\mathbb{R})$ (resp. $L^2_\varrho(\mathbb{R})$) with bounded trace provided $1 + \delta - \sigma > 0$, for all $\omega_0 \in \mathcal{D}$.

Let us now consider the closed-contour kernels $\mathcal{G}_{C\mu b}$, for all $(\mu, b) \in C$, where we use the decomposition

$$\mathcal{G}_{C\mu b} = \sum_{\ell \in \mathbb{Z}} \mathcal{G}^\ell_{C\mu b}.$$

We first suppose that $\omega_0 \in \mathbb{C}\backslash\Sigma$. By Lemma 1, Remark 20, (144) and (128), the kernel $\mathcal{G}^\ell_{C\mu b}$ can be recast as

$$\mathcal{G}^\ell_{C\mu b}(\theta,\eta;\omega_0) = \frac{(1+|\eta|)^\sigma}{Q(\theta)} \frac{\varrho(\theta)}{\varrho(\eta)} \frac{\mathfrak{I}_{0\mu}(\theta)}{a_{\mu b}(\theta)} \frac{\mathfrak{I}_{0\mu}(\eta)}{a_{\mu b}(\eta)} \mathcal{B}_{C\mu b}(\theta,\eta,\omega_0) \mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}]}(\theta)$$
$$\left\{\cos \mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\eta;\omega_0) \cos \mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\theta;\omega_0) \mathbb{1}_{[\theta,\theta^{+,\ell}_{L\mu b}]}(\eta) + \cos \mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\eta;\omega_0) \cos \mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\theta;\omega_0) \mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta]}(\eta)\right\},$$

where $\mathcal{B}_{C\mu b}(\theta,\eta,\omega_0) \in L^\infty(\mathbb{R}_\theta \times \mathbb{R}_\eta)$. Using Lemma 1, we observe that the term $\mathcal{I}_{\mu b}(\theta,\eta;\omega_0)$ can be written as

$$\mathcal{I}_{\mu b}(\theta,\eta;\omega_0) = \mathrm{i}\,\mathrm{Im}\,\omega_0 \int_\theta^\eta d\widehat{\eta} \left(\frac{qR}{b_\varphi a_{\mu b}}\right)(\widehat{\eta}) + a(\theta,\eta) + \mathrm{i}b(\theta,\eta),$$

where a, $b \in \mathbb{R}$, and $\|b\|_{L^\infty(\mathbb{R}_\theta\times\mathbb{R}_\eta)} < \infty$. Therefore, for $\eta \in [\theta, \theta^{+,\ell}_{\mu b}]$ and $\theta \in [\theta^{-,\ell}_{L\mu b}, \theta^{+,\ell}_{L\mu b}]$, using Lemma 1, we find that

$$\begin{aligned}
\cos \mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\eta;\omega_0) &\le C(\|b\|_{L^\infty(\mathbb{R}_\theta\times\mathbb{R}_\eta)}) \Bigg(\exp\left(-\mathrm{Im}\,\omega_0 \int_{\theta^{+,\ell}_{L\mu b}}^\eta d\widehat{\eta} \left(\frac{qR}{b_\varphi a_{\mu b}}\right)(\widehat{\eta})\right) \\
&\qquad + \exp\left(\mathrm{Im}\,\omega_0 \int_{\theta^{+,\ell}_{L\mu b}}^\eta d\widehat{\eta} \left(\frac{qR}{b_\varphi a_{\mu b}}\right)(\widehat{\eta})\right)\Bigg) \\
&\le C(\|b\|_{L^\infty(\mathbb{R}_\theta\times\mathbb{R}_\eta)}) \left(1 + \exp\left(|\mathrm{Im}\,\omega_0| \int_\theta^{\theta^{+,\ell}_{L\mu b}} d\widehat{\eta} \left(\frac{qR}{b_\varphi a_{\mu b}}\right)(\widehat{\eta})\right)\right) \\
&\le C(\|b\|_{L^\infty(\mathbb{R}_\theta\times\mathbb{R}_\eta)}) \left(1 + \exp\left(|\mathrm{Im}\,\omega_0| \int_{\theta^{-,\ell}_{L\mu b}}^{\theta^{+,\ell}_{L\mu b}} d\widehat{\eta} \left(\frac{qR}{b_\varphi a_{\mu b}}\right)(\widehat{\eta})\right)\right) \\
&\le C(\|b\|_{L^\infty(\mathbb{R}_\theta\times\mathbb{R}_\eta)}) \left(1 + \exp\left(|\mathrm{Im}\,\omega_0| \int_{-\theta_{L\mu b}}^{\theta_{L\mu b}} d\widehat{\eta} \left(\frac{qR}{b_\varphi a_{\mu b}}\right)(\widehat{\eta})\right)\right) \\
&< \infty.
\end{aligned} \tag{145}$$

In the same way as we have bounded $\cos \mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\eta;\omega_0)$, we can bound $\cos \mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\theta;\omega_0)$, $\cos \mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\eta;\omega_0)$, and $\cos \mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\theta;\omega_0)$ with the same bound (145). Consequently, we obtain

$$|\mathcal{G}^\ell_{C\mu b}(\theta,\eta;\omega_0)| \le C \frac{(1+|\eta|)^\sigma}{Q(\theta)} \frac{\varrho(\theta)}{\varrho(\eta)} \frac{|\mathfrak{I}_{0\mu}(\theta)|}{a_{\mu b}(\theta)} \frac{|\mathfrak{I}_{0\mu}(\eta)|}{a_{\mu b}(\eta)} \mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}]}(\theta) \mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}]}(\eta),$$

and thus, using the disjoint support property, we obtain

$$\begin{aligned}
\int_\mathbb{R}\int_\mathbb{R} d\theta d\eta\, |\mathcal{G}_{C\mu b}|^2 &= \int_\mathbb{R}\int_\mathbb{R} d\theta d\eta \left|\sum_{\ell\in\mathbb{Z}} \mathcal{G}^\ell_{C\mu b}\right|^2 = \sum_{\ell\in\mathbb{Z}} \int_\mathbb{R}\int_\mathbb{R} d\theta d\eta\, |\mathcal{G}^\ell_{C\mu b}|^2 \\
&\le \sum_{\ell\in\mathbb{Z}} \int_{\theta^{-,\ell}_{L\mu b}}^{\theta^{+,\ell}_{L\mu b}} d\theta \int_{\theta^{-,\ell}_{L\mu b}}^{\theta^{+,\ell}_{L\mu b}} d\eta |\mathbb{G}_{C\mu b}|^2 \le \int_\mathbb{R}\int_\mathbb{R} d\theta d\eta\, |\mathbb{G}_{C\mu b}|^2,
\end{aligned}$$

where

$$\mathbb{G}_{C\mu b}(\theta,\eta) = C\frac{(1+|\eta|)^{\sigma}}{Q(\theta)}\frac{\varrho(\theta)}{\varrho(\eta)}\frac{|\Im_{0\mu}(\theta)|}{a_{\mu b}(\theta)}\frac{|\Im_{0\mu}(\eta)|}{a_{\mu b}(\eta)}.$$

For $\eta,\ \theta \in \mathcal{V}(\pm\infty)$, using the previous equation, we get $|\mathbb{G}_{C\mu b}| \lesssim (1+|\theta|)^{-2+2\alpha-1/2}(1+|\eta|)^{\sigma-2\alpha-1/2} \in L^2(\mathbb{R}_\theta \times \mathbb{R}_\eta)$ as long as $\sigma/2 < \alpha < 1$. We now get for all $\eta \in \mathcal{V}(\theta^{s',\ell'}_{L\mu'b'})$ and $\theta \in \mathcal{V}(\theta^{s'',\ell''}_{L\mu''b''})$,

$$|\mathbb{G}_{C\mu b}(\theta,\eta)| \lesssim \frac{|\Im_{0\mu}(\theta)|}{a_{\mu b}(\theta)}\frac{|\Im_{0\mu}(\eta)|}{a_{\mu b}(\eta)}\frac{1}{\varpi^{1/\beta-1}(\theta)\varpi(\eta)} = \Gamma_{\mu b}(\theta)\Pi_{\mu b}(\eta).$$

Let us first look at $\Pi_{\mu b}(\eta)$. If $(\mu',b') \neq (\mu,b)$, then for all $\eta \in \mathcal{V}(\theta^{s',\ell'}_{L\mu'b'})$, we have $\Pi_{\mu b}(\eta) \lesssim (a_{\mu'b'}/|\Im_{0\mu'}|^2)^{\beta}$. Under hypothesis (**H**) and the condition $\beta > 0$, we then get $\Pi_{\mu b} \in L^2(\mathcal{V}(\theta^{s',\ell'}_{L\mu'b'}))$. Otherwise, if $(\mu',b') = (\mu,b)$, then for all $\eta \in \mathcal{V}(\theta^{s',\ell'}_{L\mu'b'})$, we have $\Pi_{\mu b}(\eta) \lesssim 1/a^{1-\beta}_{\mu'b'}/|\Im_{0\mu'}|^{2\beta-1}$. Under hypothesis (**H**) and the condition $\beta > 0$, we then get $\Pi_{\mu b} \in L^2(\mathcal{V}(\theta^{s',\ell'}_{L\mu'b'}))$. Let us note that the same type of estimates can be obtained if some pairs of limit angles have identical values, since the corresponding equilibrium closed contours have the same algebraic singularity of power minus one-half. Now, let us look at $\Gamma_{\mu b}(\theta)$. If $(\mu'',b'') \neq (\mu,b)$ then for all $\theta \in \mathcal{V}(\theta^{s'',\ell''}_{L\mu''b''})$, we have $\Gamma_{\mu b}(\theta) \lesssim (a_{\mu''b''}/|\Im_{0\mu''}|^2)^{1-\beta}$. Under the condition $\beta < 1$ and hypothesis (**H**), we then get $\Gamma_{\mu b} \in L^2(\mathcal{V}(\theta^{s'',\ell''}_{L\mu''b''}))$. Otherwise, if $(\mu'',b'') = (\mu,b)$, then for all $\theta \in \mathcal{V}(\theta^{s'',\ell''}_{L\mu''b''})$, we have $\Gamma_{\mu b}(\theta) \lesssim 1/a^{\beta}_{\mu''b''}/|\Im_{0\mu''}|^{1-2\beta}$. Under the condition $\beta < 1$ and hypothesis (**H**), we then get $\Gamma_{\mu b} \in L^2(\mathcal{V}(\theta^{s'',\ell''}_{L\mu''b''}))$. Let us note that the same type of estimates can be established if some pairs of limit angles have identical values, since the corresponding equilibrium closed contours have the same algebraic singularity of power minus one-half. Moreover for all $\eta \in \mathcal{V}(\theta^{s,\ell}_{L\mu b})$, we have $|\mathcal{G}_{C\mu b}(\eta,\eta)| \lesssim 1/a_{\mu b} \in L^1(\mathcal{V}(\theta^{s,\ell}_{L\mu b}))$, while for all $\eta \in \mathcal{V}(\pm\infty)$, $|\mathcal{G}_{C\mu b}(\eta,\eta)| \lesssim 1/(1+|\eta|^2) \in L^1(\mathcal{V}(\pm\infty))$. Therefore, we infer that $\forall \omega_0 \in \mathbb{C}\backslash\Sigma$, the operator-valued function $\omega_0 \longmapsto \mathrm{Op}(\mathcal{G}_C)(\omega_0)$ (resp. $\omega_0 \longmapsto \mathrm{Op}(\mathbb{K}_C)(\omega_0)$) constitutes analytic families of Hilbert-Schmidt operators in $L^2(\mathbb{R})$ (resp. $L^2_\varrho(\mathbb{R})$) with bounded trace. Using the Taylor expansion of the sinus functions $\sin \mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b}, \theta^{-,\ell}_{L\mu b};\omega_0)$ around a point $\omega \in \Sigma$, we obtain the Laurent series expansion of the ω_0-operator-valued functions $\mathrm{Op}(\mathcal{G}_C)(\omega_0)$ and $\mathrm{Op}(\mathbb{K}_C)(\omega_0)$. For example, we have

$$\mathrm{Op}(\mathcal{G}_C)(\omega_0) = \mathrm{Op}(\mathcal{G}_{C_{-1}})(\omega)(\omega_0-\omega)^{-1} + \sum_{k=0}^{\infty}\mathrm{Op}(\mathcal{G}_{C_k})(\omega)(\omega_0-\omega)^k. \tag{146}$$

From the above analysis, we can show that the terms $\{\mathrm{Op}(\mathcal{G}_{C_k})(\omega)\}_{k\geq -1}$, (resp. $\{\mathrm{Op}(\mathbb{K}_{C_k})(\omega)\}_{k\geq -1}$) of the Laurent series constitute families of Hilbert-Schmidt operators in $L^2(\mathbb{R})$ (resp. $L^2_\varrho(\mathbb{R})$) with bounded trace. Therefore, $\mathrm{Op}(\mathcal{G})(\omega_0) : \mathcal{D} \longmapsto \mathcal{L}(L^2)$ (resp. $\mathrm{Op}(\mathbb{K})(\omega_0) : \mathcal{D} \longmapsto \mathcal{L}(L^2_\varrho)$) are meromorphic operator-valued function on any open connected subset $\mathcal{D}$ of $\mathbb{C}$ where the coefficients of the Laurent series of $\mathrm{Op}(\mathcal{G})(\omega_0)$ (resp. $\mathrm{Op}(\mathbb{K})(\omega_0)$) — in particular the residues at the simple poles Σ — are Hilbert-Schmidt, hence compact operators on $L^2(\mathbb{R})$ (resp. $L^2_\varrho(\mathbb{R})$). □

As a consequence of Proposition 10, we obtain the following analytic Fredholm theorem for the open-and-closed-contour operator in $\mathbb{C}\backslash\Sigma$.

Theorem 5. *Let us suppose that assumptions of Lemma 1 and hypothesis (***H***) are satisfied. Let Ω be any open connected subset of $\mathbb{C}\backslash\Sigma$. Then either*

(i) *$I - \mathrm{Op}(\mathcal{G})$ is nowhere invertible in Ω, or*
(ii) *the resolvent $(I - \mathrm{Op}(\mathcal{G}))^{-1}$ exists for all $\omega_0 \in \Omega\backslash\mathcal{S}$, where $\mathcal{S}$ is a discrete subset of Ω constituted of a countable number of isolated points. In the latter case the resolvent $(I - \mathrm{Op}(\mathcal{G}))^{-1}$ extends to an operator-valued function in ω_0 which is analytic in $\Omega\backslash\mathcal{S}$, meromorphic in Ω, and the residues at the poles $\omega_0 \in \mathcal{S}$ are finite rank operators. If $\omega_0 \in \mathcal{S}$, then the equations $(I - \mathrm{Op}(\mathcal{G})(\omega_0))\phi = 0$, and $(I - \mathrm{Op}(\mathcal{G}^\star)(\omega_0))\psi = 0$ have the same number of linearly independent solutions; these are non-zero in $L^2(\mathbb{R})$ and hence almost everywhere.*

Proof. From Proposition 10, we obtain that $\mathrm{Op}(\mathcal{G})(\omega_0) : \Omega \longmapsto \mathcal{L}(L^2)$ is an analytic operator-valued function such that $\mathrm{Op}(\mathcal{G})(\omega_0)$ is compact for each $\omega_0 \in \Omega$. This, together with the analytic Fredholm theorem such as Theorem VI.14 of Ref. 89 or Theorem 1 of Ref. 97, implies Theorem 5. □

Remark 24. A theorem similar to Theorem 5 can be stated for the operator-valued function $I - \mathrm{Op}(\mathbb{K})(\omega_0) : \Omega \longmapsto \mathcal{L}(L^2_\varrho(\mathbb{R}))$, *in the Hilbert space* $L^2_\varrho(\mathbb{R})$. *Therefore, the poloidal envelope* $\phi_{10\omega n}$ *of the eigenmode belongs to* $L^2_\varrho(\mathbb{R})$, *which means that the map* $\theta \longmapsto \phi_{10\omega n}(\theta)$ *decays rapidly enough at infinity and integrable singularity is not ruled out.*

Remark 25. Let us note that the operators $\mathrm{Op}(\mathbb{K}_C)$ *and* $\mathrm{Op}(\mathcal{G}_C)$ *have semi-separable kernels (see Chapter IX*[49]*) but not-separable or degenerate kernels,*[89,69,95] *i.e., they are not defined through a sum of finite number of product of functions of* θ *alone by functions of* η *alone. Then the operator* $\mathrm{Op}(\mathcal{G}_{C_{-1}})$ *in the Laurent series (146) or the associated operator* $\mathrm{Op}(\mathbb{K}_{C_{-1}})$ *has semi-separable kernels. Therefore the ranges of the operators* $\mathrm{Op}(\mathcal{G}_{C_{-1}})$ *and* $\mathrm{Op}(\mathbb{K}_{C_{-1}})$ *are not in general finite dimensional (see Chapter IX*[49]*). Consequently, we cannot use the meromorphic Fredholm theorem (see Theorem XIII.13 in Ref. 88 or Theorem 2 in Ref. 97) to extend the resolvents* $\omega_0 \longmapsto (I - \mathrm{Op}(\mathcal{G})(\omega_0))^{-1}$ *and* $\omega_0 \longmapsto (I - \mathrm{Op}(\mathbb{K})(\omega_0))^{-1}$ *to operator-valued functions in* ω_0 *that are analytic in* $\Omega\backslash\mathcal{S}$ *and meromorphic in* Ω, *where now* Ω *is any open connected subset of* $\mathbb{C}$.

Actually, we can remove the assumption (**H**) and prove that meromorphic operator families $\mathbb{C} \ni \omega_0 \longmapsto \mathrm{Op}(\mathbb{K}_C)(\omega_0)$ are still compact but not Hilbert-Schmidt, in some Hilbert spaces. More precisely, we have the following proposition.

Proposition 11. Let us assume hypothesis of Lemma 1. Then, $\mathrm{Op}(\mathbb{K}_C)(\omega_0) : \mathbb{C}\backslash\Sigma \longmapsto \mathcal{L}(L^2)$ *is an analytic operator-valued function such that* $\mathrm{Op}(\mathbb{K}_C)(\omega_0)$ *is compact on* $L^2(\mathbb{R})$ *for each* $\omega_0 \in \mathbb{C}\backslash\Sigma$. *Moreover* $\mathrm{Op}(\mathbb{K}_C)(\omega_0) : \mathbb{C} \longmapsto \mathcal{L}(L^2)$ *is a meromorphic operator-valued function of* $\mathbb{C}$ *where the coefficients of the Laurent series of* $\mathrm{Op}(\mathbb{K}_C)(\omega_0)$ *— in particular the residues at the simple poles* Σ *— are compact operators on* $L^2(\mathbb{R})$.

Proof. Let us note that the operator $\mathrm{Op}(\mathbb{K}_C)$ can be written as

$$\mathrm{Op}(\mathbb{K}_C) = \mathrm{Op}\left(\sum_{\mu b \in C}\sum_{\ell\in\mathbb{Z}} \mathbb{K}^\ell_{C\mu b}\right) = \sum_{\mu b \in C}\sum_{\ell\in\mathbb{Z}} \mathrm{Op}\left(\mathbb{K}^\ell_{C\mu b}\right),$$

with

$$\mathbb{K}^\ell_{C\mu b}(\theta,\eta;\omega_0) = \frac{(1+|\eta|)^\sigma}{Q(\theta)}\frac{\mathfrak{J}_{0\mu}(\theta)}{a_{\mu b}(\theta)}\frac{\mathfrak{J}_{0\mu}(\eta)}{a_{\mu b}(\eta)}\mathcal{B}_{C\mu b}(\theta,\eta,\omega_0)\mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}]}(\theta)$$
$$\left\{\cos\mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\eta;\omega_0)\cos\mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\theta;\omega_0)\mathbb{1}_{[\theta,\theta^{+,\ell}_{L\mu b}]}(\eta) + \cos\mathcal{I}_{\mu b}(\theta^{-,\ell}_{L\mu b},\eta;\omega_0)\cos\mathcal{I}_{\mu b}(\theta^{+,\ell}_{L\mu b},\theta;\omega_0)\mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta]}(\eta)\right\}.$$

For every $\omega_0 \in \mathbb{C}\backslash\Sigma$, we have $\mathcal{B}_{C\mu b}(\theta,\eta,\omega_0) \in L^\infty(\mathbb{R}_\theta\times\mathbb{R}_\eta)$. Therefore, using the same type of estimate as (145), obtained in the proof of Proposition 10, we find that

$$\begin{aligned}\|\mathrm{Op}(\mathbb{K}_C)\phi\|_{L^2(\mathbb{R})} &= \left\|\sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}\mathrm{Op}\left(\mathbb{K}^\ell_{C\mu b}\right)\phi\right\|_{L^2(\mathbb{R})} \le \sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}\left\|\mathrm{Op}\left(\mathbb{K}^\ell_{C\mu b}\right)\phi\right\|_{L^2(\mathbb{R})}\\ &\le \sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}\left\|\mathrm{Op}\left(\left|\mathbb{K}^\ell_{C\mu b}\right|\right)|\phi|\right\|_{L^2(\mathbb{R})} \le \sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}\left\|T^\ell_{B\mu b}\circ T^\ell_{C\mu b}|\phi|\right\|_{L^2(\mathbb{R})}\\ &= \sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}\left\|T^\ell_{B\mu b}\circ T^\ell_{C\mu b}|\phi|\right\|_{L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])},\end{aligned} \tag{147}$$

where $T^\ell_{B\mu b}$ is the multiplication operator

$$T^\ell_{B\mu b}\varphi = \frac{|\mathfrak{J}_{0\mu}(\theta)|}{a_{\mu b}(\theta)}\frac{1}{Q(\theta)}\mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}]}(\theta)\varphi(\theta) = \mathfrak{m}(\theta)\varphi(\theta),$$

and $T^{\ell}_{C\mu b}$ is the weakly singular operator

$$T^{\ell}_{C\mu b}\varphi = \int_{\theta^{-,\ell}_{L\mu b}}^{\theta^{+,\ell}_{L\mu b}} d\eta \frac{|\mathfrak{J}_{0\mu}(\eta)|}{a_{\mu b}(\eta)} |\mathcal{B}_{C\mu b}(\theta,\eta,\omega_0)|(1+|\eta|)^{\sigma} \mathbb{1}_{[\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}]}(\eta)\varphi(\eta).$$

Let us first show that $\mathfrak{m} = |\mathfrak{J}_{0\mu}|a^{-1}_{\mu b}Q^{-1} \in L^p \cap L^{\infty}(\mathbb{R}_\theta)$ with $p \geq 1$. Indeed for $\theta \in \mathcal{V}(\pm\infty)$, we have $\mathfrak{m} \lesssim C/(1+|\theta|^2)$. In the sequel of the proof, C will denote a generic constant which will change from line to line. Let us now suppose that $\theta \in \mathcal{V}(\theta_L)$, where θ_L is any limit angle belonging to the set $\{\theta^{\pm,\ell}_{L\mu b}\}_{\mu b\in C,\ \ell\in\mathbb{Z}}$. Indeed, since the zeros of Bessel function of first kind of zeroth order J_0 are of order greater than one-half, if θ_L is a zero of $\mathfrak{J}_{0\mu}$, then there is no singularity at θ_L. If however θ_L is not a zero of $\mathfrak{J}_{0\mu}$, then the algebraic singularity of order minus one-half of $a^{-1}_{\mu b}$ at θ_L (see (134)) is compensated by Q. Therefore $\mathfrak{m} \in L^p \cap L^{\infty}(\mathbb{R}_\theta)$, with $p \geq 1$. Since $\mathfrak{m} \in L^p \cap L^{\infty}(\mathbb{R}_\theta)$, with $p \geq 1$, the operator $T^{\ell}_{B\mu b}$ is a bounded operator in $L^2(\mathbb{R})$. Moreover we easily get the estimate

$$\begin{aligned} \left\|T^{\ell}_{B\mu b}\varphi\right\|_{L^2(\mathbb{R})} &= \left\|T^{\ell}_{B\mu b}\varphi\right\|_{L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])} \\ &\leq C(1+|\theta^{-,\ell}_{L\mu b}|^2)^{-1}\|\varphi\|_{L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])} \\ &\leq C(1+|\theta^{-,\ell}_{L\mu b}|^2)^{-1}\|\varphi\|_{L^2(\mathbb{R})}. \end{aligned} \tag{148}$$

Now, we can recast the operator $T^{\ell}_{C\mu b}$ as

$$T^{\ell}_{C\mu b}\varphi = \int_{\theta^{-,\ell}_{L\mu b}}^{\theta^{+,\ell}_{L\mu b}} d\eta \frac{K(\eta,\theta,\omega_0)}{a_{\mu b}(\eta)}\varphi(\eta),$$

where the kernel $K(\eta,\theta,\omega_0)$, jointly continuous in the variable η and θ, is such that

$$\|K\|_{L^{\infty}([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])} \leq C(1+|\theta^{+,\ell}_{L\mu b}|^2)^{\sigma/2}, \tag{149}$$

for every $\omega_0 \in \mathbb{C}\backslash\Sigma$. At the neighborhood of the limit angles $\theta^{\pm,\ell}_{L\mu b}$, using (134), we have the estimate

$$\frac{1}{a_{\mu b}(\eta)} \simeq w\left(\left|\eta - \theta^{\pm,\ell}_{L\mu b}\right|\right),$$

where the weight function $w :]0,+\infty[\longmapsto \mathbb{R}$ represents the weak singularity, i.e., w is continuous and satisfies $|w(\eta)| \leq C\eta^{-1/2}$. The power minus one-half comes from the integrable algebraic singularity of order minus one-half of the closed contours $\{a_{\mu b}\}_{(\mu,b)\in C}$ (see (134)). The following is well known (see Ref. 71 or Section 9.5 of Ref. 33 and Exercises 9.19 to 9.22 of Ref. 33): let T be an integral operator with weak singularity, that is, defined by

$$(T\varphi)(t) = \int_{\Gamma} \nu(|t-\tau|)\mathfrak{B}(t,\tau)\varphi(\tau),$$

where the kernel $\mathfrak{B}$ is continuous and bounded on $\Gamma\times\Gamma$, and where the continuous weight function $\nu :]0,+\infty[\longmapsto \mathbb{R}$ satisfies $|\nu(t)| \leq Mt^{-\mathfrak{s}}$ with $0 \leq \mathfrak{s} < 1$, then T is a continuous endomorphism of $L^p(\Gamma)$ for $1 \leq p \leq \infty$, and a compact endomorphism of $L^p(\Gamma)$ whenever, $1 < p < \infty$, with Γ any compact set of $\mathbb{R}$. Therefore, the operator $T^{\ell}_{C\mu b}$ is a continuous and compact endomorphism on $L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])$. Moreover, using (149), we get

$$\begin{aligned} \left\|T^{\ell}_{C\mu b}\varphi\right\|_{L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])} &\leq C(1+|\theta^{+,\ell}_{L\mu b}|^2)^{\sigma/2}\left\|\int_{\theta^{-,\ell}_{L\mu b}}^{\theta^{+,\ell}_{L\mu b}} d\eta \frac{\varphi(\eta)}{a_{\mu b}(\eta)}\right\|_{L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])} \\ &\leq C(1+|\theta^{+,\ell}_{L\mu b}|^2)^{\sigma/2}\|\varphi\|_{L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])} \\ &\leq C(1+|\theta^{+,\ell}_{L\mu b}|^2)^{\sigma/2}\|\varphi\|_{L^2(\mathbb{R})}. \end{aligned}$$

Therefore, using the previous estimate, (148) and the Cauchy-Schwarz inequality, we obtain

$$\sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}\left\|T^{\ell}_{B\mu b}\circ T^{\ell}_{C\mu b}|\phi|\right\|_{L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])}\leq C\sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}\frac{(1+|\theta^{+,\ell}_{L\mu b}|^2)^{\sigma/2}}{(1+|\theta^{-,\ell}_{L\mu b}|^2)}\|\phi\|_{L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])}$$

$$\leq C\sum_{\mu b\in C}\left(\sum_{\ell\in\mathbb{Z}}\frac{(1+|\theta^{+,\ell}_{L\mu b}|^2)^{\sigma}}{(1+|\theta^{-,\ell}_{L\mu b}|^2)^2}\right)^{1/2}\left(\sum_{\ell\in\mathbb{Z}}\|\phi\|^2_{L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])}\right)^{1/2}$$

$$\leq C\|\phi\|_{L^2(\mathbb{R})},$$

because the cardinality of the set C is finite and

$$\sum_{\ell\in\mathbb{Z}}\frac{(1+|\theta^{+,\ell}_{L\mu b}|^2)^{\sigma}}{(1+|\theta^{-,\ell}_{L\mu b}|^2)^2}\leq C\sum_{\ell\in\mathbb{Z}}\frac{1}{1+|\ell|^2}<\infty.$$

Since $T^{\ell}_{B\mu b}$ is a bounded operator from $L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])$ onto $L^2(\mathbb{R})$ and $T^{\ell}_{C\mu b}$ is a compact operator from $L^2(\mathbb{R})$ onto $L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}])$ then $T^{\ell}_{B\mu b}\circ T^{\ell}_{C\mu b}$ is compact on $L^2(\mathbb{R})$. Moreover, since

$$\sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}\left\|T^{\ell}_{B\mu b}\circ T^{\ell}_{C\mu b}\right\|_{\mathscr{L}(L^2(\mathbb{R}))}=\sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}\left\|T^{\ell}_{B\mu b}\circ T^{\ell}_{C\mu b}\right\|_{\mathscr{L}(L^2([\theta^{-,\ell}_{L\mu b},\theta^{+,\ell}_{L\mu b}]))}<\infty,$$

the operator $\sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}T^{\ell}_{B\mu b}\circ T^{\ell}_{C\mu b}$ is compact on $L^2(\mathbb{R})$. Finally, using (147), we have

$$\|\mathrm{Op}(\mathbb{K}_C)\|_{\mathscr{L}(L^2(\mathbb{R}))}\leq\left\|\sum_{\mu b\in C}\sum_{\ell\in\mathbb{Z}}T^{\ell}_{B\mu b}\circ T^{\ell}_{C\mu b}\right\|_{\mathscr{L}(L^2(\mathbb{R}))}<\infty,$$

and thus $\mathrm{Op}(\mathbb{K}_C)$ is compact on $L^2(\mathbb{R})$. The end of the Proposition 11 can be proven as has been done for Proposition 10. □

As a consequence, we obtain the following results: Theorem 6 for the spectrum in $\mathbb{C}^+$ and without assumption (**H**); Theorem 7 for the spectrum in $\mathbb{C}\backslash\Sigma$ and without assumption (**H**).

Theorem 6. *Let us suppose that assumptions of Lemma 1 are satisfied. Let Ω be any open connected subset of $\mathbb{C}^+$. Then either*

(i) *$I-\mathrm{Op}(\mathbb{K})$ is nowhere invertible in Ω, or*
(ii) *the resolvent $(I-\mathrm{Op}(\mathbb{K}))^{-1}$ exists for all $\omega_0\in\Omega\backslash\mathscr{S}$, where $\mathscr{S}$ is a discrete subset of Ω constituted of a countable number of isolated points. In the latter case the resolvent $(I-\mathrm{Op}(\mathbb{K}))^{-1}$ is meromorphic in Ω, analytic in $\Omega\backslash\mathscr{S}$, and the residues at the poles are finite rank operators. If $\omega_0\in\mathscr{S}$, then the equations $(I-\mathrm{Op}(\mathbb{K})(\omega_0))\phi=0$, and $(I-\mathrm{Op}(\mathbb{K})^{\star}(\omega_0))\psi=0$ have the same number of linearly independent solutions; these are not zero in $L^2(\mathbb{R})$ and hence almost everywhere. Moreover the poles of $(I-\mathrm{Op}(\mathbb{K})(\omega_0,x))^{-1}$ in the ω_0-complex plane, depend continuously on x and can appear and disappear only at the boundary of Ω.*

Proof. From Propositions 8 and 11, we find that $\mathrm{Op}(\mathbb{K})(\omega_0):\Omega\longmapsto\mathscr{L}(L^2)$ is an analytic operator-valued function such that $\mathrm{Op}(\mathbb{K})(\omega_0)$ is compact for each $\omega_0\in\Omega$. This, together with the analytic Fredholm theorem such as Theorem VI.14 of Ref. 89, implies Theorem 6. The last assertion of Theorem 6 is a consequence of Theorem 3 of Ref. 96 and the fact that $\mathrm{Op}(\mathbb{K})(\omega_0,x)$ is a family of compact operator jointly continuous in (ω_0,x) for each $(\omega_0,x)\in\Omega\times[x_{\min},x_{\max}]$. □

Theorem 7. *Let us suppose that assumptions of Lemma 1 are satisfied. Let Ω be any open connected subset of $\mathbb{C}\backslash\Sigma$. Then either*

(i) *$I-\mathrm{Op}(\mathcal{K})$ is nowhere invertible in Ω, or*

(ii) *the resolvent $(I - \mathrm{Op}(\mathcal{K}))^{-1}$ exists for all $\omega_0 \in \Omega\backslash\mathscr{S}$, where $\mathscr{S}$ is a discrete subset of Ω constituted of a countable number of isolated points. In the latter case the resolvent $(I - \mathrm{Op}(\mathcal{K}))^{-1}$ extends to an operator-valued function in ω_0 that is analytic in $\Omega\backslash\mathscr{S}$, meromorphic in Ω, and the residues at the poles $\omega_0 \in \mathscr{S}$ are finite rank operators. If $\omega_0 \in \mathscr{S}$, then the equations $(I - \mathrm{Op}(\mathcal{K})(\omega_0))\phi = 0$, $(I - \mathrm{Op}(\mathcal{K})^\star(\omega_0))\psi = 0$ have the same number of linearly independent solutions; these are non-zero in $L^2(\mathbb{R})$ and hence almost everywhere.*

Proof. First we show that $\mathrm{Op}(\mathcal{K})(\omega_0) : \Omega \longmapsto \mathscr{L}(L^2)$ is an analytic operator-valued function such that $\mathrm{Op}(\mathcal{K})(\omega_0)$ is compact for each $\omega_0 \in \Omega$. This follows on the one hand from Propositions 8 and 9, and on the other hand from Proposition 11, where we can substitute $\mathcal{K}_C$ to $\mathbb{K}_C$ by keeping a similar proof. This, together with the analytic Fredholm theorem such as Theorem VI.14 of Ref. 89, implies Theorem 7. □

Remark 26. A theorem similar to Theorem 7 can be stated for the operator-valued function $I - \mathrm{Op}(\mathbb{K})(\omega_0) : \Omega \longmapsto \mathscr{L}(L^2_\kappa(\mathbb{R}))$, in the Hilbert space $L^2_\kappa(\mathbb{R})$.

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APPENDIX A: GLOSSARY OF MAIN NOTATION

$\mathbf{r}$: Coordinates of three-dimensional physical space.
$v_\parallel$, $\xi_\parallel$: Parallel velocity coordinates.
$(\mathbf{r}, v_\parallel)$, $(\mathbf{r}, \xi_\parallel)$: Four-dimensional phase-space.
$(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$: Toroidal vector basis.
(r, θ, φ) : toroidal coordinates.
(x, η, α) : Field-aligned coordinates.
$q(r)$: Safety factor $q = r b_\varphi/(R b_\theta)$; here, $q(r_0)$ rational $\Leftrightarrow$ r_0 is a rational magnetic flux surface.
r_0 : Constant radius denoting a reference rational magnetic flux surface.
r, x, q : Radial variables, $x = r - r_0$.
$[r_{\min}, r_{\max}]$: Radial domain.
$[x_{\min}, x_{\max}]$: Radial domain.
$[q_{\min}, q_{\max}]$: Radial domain.
$s(r)$: Shear parameter $s = q' r/q$.
a : Minor radius of the torus.
R_0 : Major radius of the torus.
$\mathbf{R}$: $\mathbf{R} = (R_0 + r\cos\theta)(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta)$ radius vector.
R : Euclidean norm of $\mathbf{R}$.
$\mathbf{B}$: Magnetic field.
B : Euclidean norm of $\mathbf{B}$.
B_θ, B_φ : Respectively, the poloidal and toroidal component of the magnetic field $\mathbf{B}$.
$\mathbf{A}$: Vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$.
$\mathbf{b}$: Unit vector tangent to the magnetic field line.
b_θ, b_φ : Respectively, the poloidal and toroidal component of the vector $\mathbf{b}$.
$B_\parallel$: $B_\parallel = \mathbf{b} \cdot \mathbf{B}$, parallel component of the magnetic field $\mathbf{B}$.

$\partial_\parallel$: $\partial_\parallel = b \cdot \nabla$, parallel gradient.

$\nabla_\perp$: $\nabla_\perp = (I - \mathbf{b} \otimes \mathbf{b})\nabla = -\mathbf{b} \times (\mathbf{b} \times \nabla)$, perpendicular gradient.

$\boldsymbol{\kappa}$: $\boldsymbol{\kappa} = \mathbf{b} \cdot \nabla \mathbf{b} = -\mathbf{b} \times \nabla \times \mathbf{b}$, curvature vector.

q_i : Ion charge.

m_i : Ion mass.

$v_{th,i}$: Ion thermal velocity.

T_{i0} : Ion temperature.

n_{i0} : Ion density.

Ω_i : $\Omega_i = qB/m$, ion cyclotron frequency.

ρ_i : $\rho_i = v_{th,i}/\Omega_i$ ion Larmor radius.

$v_\perp$: Modulus of perpendicular velocity.

μ : $\mu = m_i v_\perp^2/(2B)$, magnetic moment, adiabatic invariant, label, index.

b : Contour (or bag) index.

ω : Time frequency, global eigenfrequency.

n : Toroidal wavenumber.

m : Poloidal wavenumber.

$k_\parallel$: $k_\perp \simeq 1/(qR_0)$, parallel wavenumber.

$k_\perp$: $k_\perp \simeq n/a$, perpendicular wavenumber.

γ : $\gamma \in (0,1)$.

ϵ : Small parameter, $\epsilon = 1/n$.

ϵ_a : Small parameter, $\epsilon_a = a/R_0$.

ϵ_k : Small parameter, $\epsilon_k = k_\parallel/k_\perp$.

ϵ_ω : Small parameter, $\epsilon_\omega = \omega/\Omega_i$.

$\rho^\star$: Small parameter, $\rho^\star = \rho_i/a$.

$\mathcal{O}, \mathcal{C}, \mathscr{C}$: Set of respectively open, closed, and all contours.

J_0 : Bessel function of first kind of order zero.

$\mathcal{J}_\mu, \mathfrak{J}_\mu, \mathfrak{J}_{0\mu}$: Gyroaverage operators.

$\theta_k(x), \theta_{k1}(x), \theta_{k0}, \theta_{k0,T}$: Ballooning angles.

Θ_k : $\Theta_k = -\mathrm{i}/n\partial_q = -\mathrm{i}/(q'n)\partial_x$, radial differential operator.

$\theta_{L\mu b}, \theta_{L\mu b}^\pm, \theta_{L\mu b}^{\pm,\ell}$: Limit angles, $\theta_{L\mu b}^{\pm,\ell} = \pm\theta_{L\mu b}(r) + 2\pi\ell$.

$\mathcal{A}_{\mu b}$: Constant bag height.

$v_{\mu b}^\pm, \xi_{\mu b}^\pm$: Three-dimensional contours (level lines) of the four-dimensional phase-space $(\mathbf{r}, v_\parallel)$, and $(\mathbf{r}, \xi_\parallel)$, respectively.

$a_{\mu b}^\pm, a_{\mu b}^\circ$: Equilibrium contours.

$w_{\mu b}^\pm, w_{\mu b\omega n}^\pm$: First-order perturbation of the contours.

ϕ_0 : Equilibrium electrical potential.

$\phi_1, \phi_{1\omega n}$: First-order perturbation of the electrical potential.

$H_{\mu b}^\pm$: Hamiltonian associated to the equilibrium contours, $H_{\mu b}^\pm = a_{\mu b}^{\pm 2}/2 + \mu B/m_i$.

$h_{\mu b\omega n}^\pm$: Hamiltonian associated to the first-order perturbation, $h_{\mu b\omega n}^\pm = w_{\mu b\omega n}^\pm a_{\mu b}^\pm + q_i \mathfrak{J}_\mu \phi_{1\omega n}/m_i$.

$\omega_0(x, \theta_{k0})$: Local eigenfrequency.

$\left\langle \mathcal{L}_{\mathscr{C}\omega n}^\circ \right\rangle$: Linear integral operator of zeroth order.

$\mathbb{K}_\mathcal{O}, \mathcal{K}_\mathcal{O}, \mathcal{G}_\mathcal{O}$: Kernels of integral operator for open contours.

$\mathbb{K}_\mathcal{C}, \mathcal{K}_\mathcal{C}, \mathcal{G}_\mathcal{C}$: Kernels of integral operator for closed contours.

f, f_μ : Gyrokinetic distribution function in the variables $(\mathbf{r}, v_\parallel)$.

$\mathfrak{f}, \mathfrak{f}_\mu$: Gyrokinetic distribution function in the variables $(\mathbf{r}, \xi_\parallel)$.

$\mathbf{F} = (\mathbf{F}_\mathbf{r}, F_{v_\parallel})$: Four-dimensional force vector-field in the variables $(\mathbf{r}, v_\parallel)$.

$\mathfrak{F} = (\mathfrak{F}_\mathbf{r}, \mathfrak{F}_{\xi_\parallel})$: Four-dimensional force vector-field in the variables $(\mathbf{r}, \xi_\parallel)$.

$\mathcal{H}$: Hamiltonian, $\mathcal{H} = m_i v_\parallel^2/2 + \mu B + q_i \mathcal{J}_\mu \phi$.

$\mathbf{B}^*$: $\mathbf{B}^* = \mathbf{B} + m_i v_\parallel \nabla \times \mathbf{b}/q_i$.

$B_\parallel^*$: $B_\parallel^* = \mathbf{B}^* \cdot \mathbf{b}$.

Υ : Heaviside function.

L^2 : Hilbert space of square summable function.

L_κ^2 : Hilbert space of function g, such that $g\kappa$ is square summable with $\kappa(\eta) = (1 + |\eta|^2)^\alpha$, $\alpha \in ([1-\delta]/4 + \sigma/2, [3+\delta]/4)$, $\sigma \in \{0,1\}$, $\delta \in \{0,1\}$.

L_ϱ^2 : Hilbert space of function g, such that $g\varrho$ is square summable with $\varrho = \kappa\varpi$, $\varpi = (\sum_{\mu b \in \mathscr{C}} \mathcal{A}_{\mu b} \mathfrak{J}_{0\mu}^2/a_{\mu b})^\beta \in L_{\mathrm{loc}}^\gamma(\mathbb{R})$, $\beta \in (0,1)$, $\gamma \in (0, 2/\beta)$.

APPENDIX B: RIGOROUS DERIVATION OF THE GYROKINETIC-WATERBAG EQUATIONS

As pointed out in Remark 2, of Sec. II B 2, the variables $(\mathbf{r},\xi_\parallel)$ are well suited for applying the waterbag reduction concept. Here,

$$\xi_\parallel = \int^{v_\parallel} dv_\parallel J(\mathbf{r},v_\parallel), \tag{B1}$$

and $J(\mathbf{r},v_\parallel) = \mathbf{B}\cdot\mathbf{b} + v_\parallel(m_i/q_i)\mathbf{b}\cdot\nabla\times\mathbf{b}$. Let us thus rewrite the gyrokinetic-Vlasov equation (3) in this new set of variables. We introduce the new distribution function $\mathfrak{f} = \mathfrak{f}(t,\mathbf{r},\xi_\parallel,\mu)$ such that $\mathfrak{f}(t,\mathbf{r},\xi_\parallel,\mu) = f(t,\mathbf{r},v_\parallel,\mu)$. Using the change of variables $(\mathbf{r},\xi_\parallel)\leftrightarrow(\mathbf{r},v_\parallel)$ and the chain rule, we easily get the transformations

$$(\nabla_\mathbf{r},\partial_{v_\parallel}) \longrightarrow (\nabla_\mathbf{r} + \nabla_\mathbf{r}\xi_\parallel\partial_{\xi_\parallel}, \partial_{v_\parallel}\xi_\parallel\partial_{\xi_\parallel}), \qquad (\nabla_\mathbf{r},\partial_{\xi_\parallel}) \longrightarrow (\nabla_\mathbf{r} + \nabla_\mathbf{r}v_\parallel\partial_{v_\parallel}, \partial_{\xi_\parallel}v_\parallel\partial_{v_\parallel}).$$

This leads to the new gyrokinetic-Vlasov equation for $\mathfrak{f}$,

$$\partial_t\mathfrak{f} + \mathfrak{F}_\mathbf{r}\cdot\nabla_\mathbf{r}\mathfrak{f} + \mathfrak{F}_{\xi_\parallel}\partial_{\xi_\parallel}\mathfrak{f} = 0, \tag{B2}$$

with

$$\mathfrak{F}_\mathbf{r} = \mathbf{F}_\mathbf{r}, \qquad \mathfrak{F}_{\xi_\parallel} = \mathbf{F}_\mathbf{r}\cdot\nabla_\mathbf{r}\xi_\parallel + \mathrm{F}_{v_\parallel}\partial_{v_\parallel}\xi_\parallel.$$

It can be easily checked that the new force vector-field $\mathfrak{F} = (\mathfrak{F}_\mathbf{r},\mathfrak{F}_{\xi_\parallel})$ is divergence-free, i.e., $\nabla_{\mathbf{r},\xi_\parallel}\cdot\mathfrak{F} = \nabla_\mathbf{r}\cdot\mathfrak{F}_\mathbf{r} + \partial_{\xi_\parallel}\mathfrak{F}_{\xi_\parallel} = 0$. Therefore the flow $(\mathbf{r},\xi_\parallel)\longmapsto(\mathbf{R}(t),\Xi_\parallel(t))$ generated by the force vector-field $\mathfrak{F}$ (solution of $d_t\mathbf{R} = \mathfrak{F}_\mathbf{r}$, $d_t\Xi_\parallel = \mathfrak{F}_{\xi_\parallel}$, with the initial conditions $(\mathbf{R}(0),\Xi_\parallel(0)) = (\mathbf{r},\xi_\parallel)$) defines a volume-preserving map, i.e., the following Liouville theorem:

$$\frac{d}{dt}\int_{\Omega(t)} d\mathbf{r}d\xi_\parallel = 0$$

is satisfied. Here, $\Omega(t)$ is the image of any bounded phase-space volume element $\Omega(0)$ from the Lagrangian flow $(\mathbf{R}(t),\Xi_\parallel(t))$ induced by the force field $\mathfrak{F}$.

Therefore, for every adiabatic invariant μ, we can consider $2\mathcal{N}$ non-closed single-valued contours $\{\xi^\pm_{\mu b}(t,\mathbf{r})\}_{b\le\mathcal{N}}$ of the $(\mathbf{r},\xi_\parallel)$-phase space ordered such that $\ldots<\xi^-_{\mu\, b+1}<\xi^-_{\mu b}<\cdots\le 0\le\cdots<\xi^+_{\mu b}<\xi^+_{\mu\, b+1}<\cdots$, and strictly positive real numbers $\{\mathcal{A}_{\mu b}\}_{b\le\mathcal{N}}$, called the bag heights. From the Liouville theorem in the phase-space $(\mathbf{r},\xi_\parallel)$, we know that for every couple (μ,b),

$$\frac{d}{dt}\int \mathcal{A}_{\mu b}(\xi^+_{\mu b} - \xi^-_{\mu b})d\mathbf{r} = 0.$$

We observe that the distribution $\mathfrak{f}$ reads

$$\mathfrak{f}(t,\mathbf{r},\xi_\parallel,\mu) = \int_{\mathbb{R}^+}\mathfrak{f}_\nu(t,\mathbf{r},\xi_\parallel)\delta_\nu(\mu)\mathfrak{m}(d\nu),$$

where $\mathfrak{m}$ is a probability measure on $\mathbb{R}^+$ and where the smooth functions $\mathfrak{f}_\mu$ still satisfy the gyrokinetic-Vlasov equation (B2). We can now take for $\mathfrak{f}_\mu$ the waterbag distribution function,

$$\mathfrak{f}_\mu(t,\mathbf{r},\xi_\parallel) = \sum_{b=1}^{\mathcal{N}}\mathcal{A}_{\mu b}\left(\Upsilon\left(\xi^+_{\mu b}(t,\mathbf{r}) - \xi_\parallel\right) - \Upsilon\left(\xi^-_{\mu b}(t,\mathbf{r}) - \xi_\parallel\right)\right). \tag{B3}$$

As long as the contours $\xi^\pm_{\mu b}$ are smooth, single-valued, and do not cross, the function (B3) is an exact weak solution of the gyrokinetic-Vlasov equation (B2) in the sense of distribution theory, if and only if the following gyrowaterbag equations in advective form are satisfied:

$$\partial_t\xi^\pm_{\mu b} + \mathfrak{F}_\mathbf{r}(\mathbf{r},\xi^\pm_{\mu b})\cdot\nabla\xi^\pm_{\mu b} = \mathfrak{F}_{\xi_\parallel}(\mathbf{r},\xi^\pm_{\mu b}). \tag{B4}$$

Furthermore (B1) is equivalent to

$$\xi_\parallel(\mathbf{r},v_\parallel) = v_\parallel B_\parallel(1+\Lambda_\parallel v_\parallel), \quad \text{where} \quad \Lambda_\parallel = \frac{m_i}{q_i}\frac{\mathbf{b}.\nabla\times\mathbf{b}}{2B_\parallel}.$$

Solving the previous quadratic equation, we obtain the solution,

$$v_\parallel(\mathbf{r},\xi_\parallel) = \frac{-1+\sqrt{1+4\xi_\parallel\Lambda_\parallel/B_\parallel}}{2\Lambda_\parallel},$$

which leads to the definition of the new contours $v^{\pm}_{\mu b}$ of the phase-space $(\mathbf{r}, v_{\|})$ as

$$v^{\pm}_{\mu b} := v_{\|}(\mathbf{r}, \xi^{\pm}_{\mu b}), \quad \Longleftrightarrow \quad \xi^{\pm}_{\mu b} = v^{\pm}_{\mu b} B_{\|}(1 + \Lambda_{\|} v^{\pm}_{\mu b}).$$

Using the previous definition in (B4), we get, after some algebra,

$$\partial_t \xi^{\pm}_{\mu b} + \nabla \cdot \left(\frac{1}{m_i q_i} \left(q_i \mathbf{A} + m_i v_{\|}(\mathbf{r}, \xi^{\pm}_{\mu b}) \mathbf{b} \right) \times \nabla \mathcal{H}(\mathbf{r}, v_{\|}(\mathbf{r}, \xi^{\pm}_{\mu b})) \right) = 0.$$

This is exactly the gyrokinetic-waterbag equations (15) of Sec. II B 2 and thus definitely ensures the validity of the gyrokinetic-waterbag equations (15).

Remark 27. Using $|\mathbf{b}.\nabla \times \mathbf{b}| \simeq 1/R$ (see formula (19) of Sec. II C), we find that $\xi_{\|}\Lambda_{\|}/B_{\|} = \mathcal{O}(\epsilon_\omega) \ll 1$ and thus, at first order in ϵ_ω (see Sec. II D for the definition of ϵ_ω), we obtain $v_{\|} = \xi_{\|}/B_{\|} + \mathcal{O}(\epsilon_\omega)$, which leads to

$$v^{\pm}_{\mu b} = \frac{\xi^{\pm}_{\mu b}}{B_{\|}} + \mathcal{O}(\epsilon_\omega).$$

APPENDIX C: LINEARIZATION OF THE GYROKINETIC-WATERBAG EQUATIONS

Here, we explain in detail how to obtain the linearized gyrowaterbag equations (20) and (22) of Sec. III A from the nonlinear gyrowaterbag equation (15) of Sec. II B 2. Using the decomposition

$$\phi(t, \mathbf{r}) = \phi_0(r, \theta) + \phi_1(t, \mathbf{r}), \qquad (|\phi_1| \ll 1),$$
$$v^{\pm}_{\mu b}(t, \mathbf{r}) = a^{\pm}_{\mu b}(r, \theta) + w^{\pm}_{\mu b}(t, \mathbf{r}), \qquad (|w^{\pm}_{\mu b}| \ll 1),$$

as an (r, θ)-dependent equilibrium plus a $(t, \mathbf{r})$-dependent small perturbation, we obtain at zeroth order (with respect to the perturbation terms) the equation for the steady equilibrium state

$$\begin{aligned} a^{\pm}_{\mu b} \left[1 + \frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \cdot \nabla \times \mathbf{b} \right] \mathbf{b} \cdot \nabla a^{\pm}_{\mu b} &+ \left(\frac{\mu}{q_i} \frac{\mathbf{b} \times \nabla B}{B} + \frac{a^{\pm 2}_{\mu b}}{\Omega_i} \mathbf{b} \times \boldsymbol{\kappa} \right) \cdot \nabla a^{\pm}_{\mu b} \\ &+ \frac{\mu}{m_i} \nabla B \cdot \left(\mathbf{b} \left[1 + \frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \cdot \nabla \times \mathbf{b} \right] + \frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \times \boldsymbol{\kappa} \right) \\ &+ \frac{q_i}{m_i} \nabla \mathcal{J}_\mu \phi_0 \cdot \left(\left[1 + \frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \cdot \nabla \times \mathbf{b} \right] \mathbf{b} + \frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \times \boldsymbol{\kappa} + \frac{1}{\Omega_i} \nabla a^{\pm}_{\mu b} \times \mathbf{b} \right) = 0, \end{aligned}$$

and at first order the equation for the unsteady small perturbation

$$\begin{aligned} \partial_t \left(w^{\pm}_{\mu b} \left[1 + \frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \cdot \nabla \times \mathbf{b} \right] \right) &+ \frac{q_i}{m_i} \mathbf{b} \cdot \nabla \mathcal{J}_\mu \phi_1 \left[1 + \frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \cdot \nabla \times \mathbf{b} \right] \\ &+ \frac{\mathbf{b} \times \nabla \mathcal{J}_\mu \phi_1}{B} \cdot \nabla a^{\pm}_{\mu b} + \frac{q_i}{m_i} \frac{a^{\pm}_{\mu b}}{\Omega_i} (\mathbf{b} \times \boldsymbol{\kappa}) \cdot \nabla \mathcal{J}_\mu \phi_1 \\ + a^{\pm}_{\mu b} \mathbf{b} \cdot \nabla w^{\pm}_{\mu b} \left[1 + \frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \cdot \nabla \times \mathbf{b} \right] &+ w^{\pm}_{\mu b} \left(\mathbf{b} \cdot \nabla a^{\pm}_{\mu b} + \frac{1}{\Omega_i} (\mathbf{b} \cdot \nabla \times \mathbf{b}) \mathbf{b} \cdot \nabla \left[\frac{a^{\pm 2}_{\mu b}}{2} + \frac{\mu}{m_i} B + \frac{q_i}{m_i} \mathcal{J}_\mu \phi_0 \right] \right) \\ + \left(\frac{a^{\pm 2}_{\mu b}}{\Omega_i} \mathbf{b} \times \boldsymbol{\kappa} + \frac{\mu}{q_i} \frac{\mathbf{b} \times \nabla B}{B} + \frac{\mathbf{b} \times \nabla \mathcal{J}_\mu \phi_0}{B} \right) \cdot \nabla w^{\pm}_{\mu b} &+ w^{\pm}_{\mu b} \left(\frac{\nabla (a^{\pm 2}_{\mu b})}{\Omega_i} + \frac{\mu}{q_i} \frac{\nabla B}{B} + \frac{\nabla \mathcal{J}_\mu \phi_0}{B} \right) \cdot \mathbf{b} \times \boldsymbol{\kappa} = 0. \end{aligned}$$

Using the approximation (D3) and the same arguments as leading to (D6) in Appendix D, we obtain

$$\mathbf{b} \times \nabla B / B \simeq \mathbf{b} \times \boldsymbol{\kappa},$$
$$\frac{a^{\pm}_{\mu b}}{\Omega_i} \mathbf{b} \cdot \nabla \times \mathbf{b} = \mathcal{O}(\epsilon_\omega), \qquad \frac{1}{\Omega_i} (\mathbf{b} \cdot \nabla \times \mathbf{b}) \mathbf{b} \cdot \nabla \left(\frac{a^{\pm 2}_{\mu b}}{2} + \frac{\mu}{m_i} B + \frac{q_i}{m_i} \mathcal{J}_\mu \phi_0 \right) = \mathcal{O}\left(\mathbf{b} \cdot \nabla a^{\pm}_{\mu b} \epsilon_\omega \right).$$

Using these estimates, we may neglect the bracket terms (except the unit term) in the previous equations and obtain the linearized Equations (20) and (22).

APPENDIX D: SOME APPROXIMATIONS RELATED TO THE TOROIDAL GEOMETRY

The magnetic field line is locally curved with a local radius-of-curvature vector $\mathbf{R}_c$. The Euclidean norm $R_c = |\mathbf{R}_c|$ is the radius of local curvature of the magnetic field line, while $\mathbf{N} = -\mathbf{R}_c/R_c$ is the unit vector in the direction of centrifugal force. Since any unit norm vector field satisfies $\mathbf{b}\cdot\nabla\mathbf{b} = -\mathbf{b}\times\nabla\times\mathbf{b}$, we obtain

$$\boldsymbol{\kappa} := \frac{\mathbf{N}}{R_c} \equiv \mathbf{b}\cdot\nabla\mathbf{b} = -\mathbf{b}\times\nabla\times\mathbf{b} = -\mathbf{b}\times\left[\frac{1}{B}\nabla\times\mathbf{B} - \mathbf{B}\times\nabla\left(\frac{1}{B}\right)\right] = \frac{\mu_0}{B^2}\mathbf{J}\times\mathbf{B} + \frac{\nabla_\perp B}{B}, \tag{D1}$$

where we have used the Faraday law $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$. For a scalar-pressure equilibrium, we have $\mathbf{J}\times\mathbf{B} = \nabla P$, where P is the plasma pressure. Thus, using (D1) and noting that $\mathbf{b}\cdot\nabla P = 0$, we obtain

$$\boldsymbol{\kappa} = \mu_0\frac{\nabla P}{B^2} + \frac{\nabla_\perp B}{B} = \frac{\mu_0}{B^2}\nabla_\perp\left(P + \frac{B^2}{2\mu_0}\right). \tag{D2}$$

In a system with low $\beta := 2\mu_0 P/B^2$, i.e., for β of order ϵ_a^2, we can use (D2), to determine the magnetic field line curvature vector $\boldsymbol{\kappa}$ as

$$\boldsymbol{\kappa} = \frac{\nabla_\perp B}{B} + O(\epsilon_a^2/a) \simeq \frac{\nabla_\perp B}{B}. \tag{D3}$$

By (D1), the approximation (D3) is equivalent to neglecting the diamagnetic current $\mathbf{J}_\perp$. Replacing $\mathbf{R}_c$ by $\mathbf{R}$, using definition of the magnetic field (18) (see Sec. II C), we obtain, after straightforward calculations,

$$\frac{\nabla_\perp B}{B} = -\frac{\mathbf{R}}{R^2} + O(\epsilon_a^2/a) \quad \text{and} \quad \mathbf{b}\times\frac{\nabla B}{B} = \mathbf{b}\times\frac{\nabla_\perp B}{B} = -\mathbf{b}\times\frac{\mathbf{R}}{R^2} + O(\epsilon_a^3/a). \tag{D4}$$

Finally, using (D3) and (D4) we have

$$\boldsymbol{\kappa} = -\frac{\mathbf{R}}{R^2} + O(\epsilon_a^2/a) \simeq -\frac{\mathbf{R}}{R^2}. \tag{D5}$$

A low-β regime (i.e., $\beta = \epsilon_a^2$) means that the plasma pressure does not play an important role in equilibria and instabilities. The low β approximation (D3) is commonly used in nonlinear gyrokinetic simulations, such as the GYSELA code.[54,40]

Now, recalling that, by (19), we have $|\mathbf{b}.\nabla\times\mathbf{b}| \simeq 1/R$, we obtain

$$\frac{B_\parallel^*}{B} = 1 + \frac{m_i v_\parallel}{q_i B}\mathbf{b}\cdot\nabla\times\mathbf{b} = 1 + O\left(\frac{v_\parallel}{\Omega_i R}\right) = 1 + O\left(\frac{k_\parallel v_\parallel}{\Omega_i}\right) = 1 + O(\epsilon_\omega) \simeq 1, \tag{D6}$$

where $\epsilon_\omega = \bar{\omega}/\Omega_i \ll 1$ and $\bar{\omega} = k_\parallel v_{th,i}$. Approximation (D6) means that the Liouville theorem $\partial_t B_\parallel^* + \nabla_{\mathbf{r}}\cdot(B_\parallel^*\mathbf{F}_{\mathbf{r}}) + \partial_{v_\parallel}(B_\parallel^* \mathrm{F}_\parallel) = 0$, which ensures the equivalence between the conservative and advective forms of the Vlasov equation (3) and energy conservation is not exactly satisfied. Preservation of the Liouville theorem is important for long-time nonlinear simulations. Indeed preservation of conservation laws in the nonlinear stage is crucial for numerical stability and for avoiding spurious effects. Let us note that approximation (D6) is used in nonlinear gyrokinetic codes.[54] In addition, linearization of the gyrokinetic-Vlasov equations (which is the starting point for the eigenvalue problem analysis and for the microinstablilities study) already leads to the loss of all nonlinear conservation laws. We observed, in Sections III B and III C, that solving equilibrium and first-order equations allows recovering conservation laws of the Hamiltonian associated to the contours. This, of course, only to relevant order. Let us note that approximation (D6) is commonly used in the study of eigenvalue problems for the characterization of kinetic microinstabilities (e.g., Refs. 99 and 98).

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