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Regularity of weak solutions for the relativistic Vlasov–Maxwell system

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Abstract. We investigate the regularity of weak solutions of the relativistic Vlasov–Maxwell system by using Fourier analysis and the smoothing effect of low velocity particles. This smoothing effect has been used by several authors (see Glassey and Strauss 1986; Klainerman and Staffilani, 2002) for proving existence and uniqueness of $\mathscr{C}^1$-regular solutions of the Vlasov–Maxwell system. This smoothing mechanism has also been used to study the regularity of solutions for a kinetic transport equation coupled with a wave equation (see Bouchut, Golse and Pallard 2004). Under the same assumptions as in the paper "Nonresonant smoothing for coupled wave + transport equations and the Vlasov–Maxwell system", *Rev. Mat. Iberoamericana* **20** (2004) 865–892, by Bouchut, Golse and Pallard, we prove a slightly better regularity for the electromagnetic field than the one showed in the latter paper. Namely, we prove that the electromagnetic field belongs to $H^s_{\mathrm{loc}}(\mathbb{R}^\star_+ \times \mathbb{R}^3)$, with $s = 6/(13+\sqrt{142})$.

Keywords: Relativistic Vlasov–Maxwell system; wave equation; Fourier analysis.

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1. Introduction

The dimensionless relativistic Vlasov–Maxwell system reads,

$$\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_p f = 0, \tag{1.1}$$

$$\partial_t E - \nabla \times B = -j, \quad \partial_t B + \nabla \times E = 0, \tag{1.2}$$

$$\nabla \cdot E = \rho, \quad \nabla \cdot B = 0, \tag{1.3}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, $p \in \mathbb{R}^3$, and $v = p/\sqrt{1+|p|^2}$ represent time, position, momentum and velocity of particles, respectively. The distribution function of particles $f = f(t,x,p)$ satisfies the Vlasov equation (1.1) with acceleration given by the Lorentz force $F_L = E + v \times B$, while the electromagnetic field $E = E(t,x)$ and $B = B(t,x)$ satisfies Maxwell's equations (1.2) and (1.3). The coupling between the Vlasov equation and Maxwell's equations occurs through the source terms of Maxwell's equations, which are the charge density $\rho = \rho(t,x)$ and the current density $j = j(t,x)$. These densities are defined as the first p-moments of the phase-space density of particles f, namely,

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,dp, \quad j(t,x) = \int_{\mathbb{R}^3} vf(t,x,p)\,dp. \tag{1.4}$$

The initial value problem associated to the system (1.1)–(1.4) requires initial conditions given by,

$$f(0,x,p) = f_0(x,p) \geq 0, \tag{1.5}$$

$$E(0,x) = E_0(x), \quad B(0,x) = B_0(x), \quad \nabla \cdot E_0 = \rho_0 = \int_{\mathbb{R}^3} f_0\,dp, \quad \nabla \cdot B_0 = 0. \tag{1.6}$$

In addition for the well-posedness of Maxwell's equations (1.2) and (1.3), the densities of charge ρ and current j must satisfy a compatibility condition given by the charge conservation law,

$$\partial_t \rho + \nabla \cdot j = 0. \tag{1.7}$$

This continuity equation is automatically satisfied if the Vlasov equation (1.1) is satisfied since it can be recovered by integration in momentum variable of the Vlasov equation.

Global existence and uniqueness of classical smooth solutions to this initial value problem has been considered by many authors, but it still remains an open problem in three dimensions. For the global well-posedness of classical solutions, standard regularity for initial data (1.5) and (1.6) is $f_0 \in \mathscr{C}_c^1(\mathbb{R}^6)$ — set of continuously differentiable functions with compact support in phase-space — and E_0, $B_0 \in \mathscr{C}^2(\mathbb{R}^3)$. Local existence and uniqueness of classical solutions for smooth and compactly supported initial data has been proved in [21]. These solutions can be extended globally in time as long as the momentum support remains bounded. Such control of the momentum support is achieved for initial data, which are small [23], or nearly neutral [16], or close to spherically symmetry [36]. In [22], the global existence and uniqueness of classical solutions has been proved under the weaker assumption that the macroscopic kinetic energy density is bounded in time and space. Recently, other refined existence criteria related to some integrability properties of the macroscopic kinetic energy density have been established in [34, 35, 29, 31, 32]. Different approaches of the Glassey–Strauss theorem [21] were recently developed: in [28], the authors used intensively Fourier or harmonic analysis [40], while in [9] the authors used a "kinetic formulation of Maxwell's equations" where Maxwell's equations can be replaced by a single scalar wave equation for a scalar potential depending of course on time and position but also momentum. For a phase-space dimension lower than six, unique global classical solutions exist for general initial data [17–20]. Finally, we mention that many results concerning the Cauchy problem for kinetic equations, and especially for the relativistic Vlasov–Maxwell system, are reviewed in [15, 11].

Until now there is no evidence that generic classical solutions in three dimensions would develop singularities in a finite time. Nevertheless proving such a conjecture remains a challenging open problem. To obtain global-in-time solutions of Vlasov–Maxwell systems, DiPerna and Lions [13] have considered a weaker notion of solutions, which were revisited in [37]. In [13], the crucial issue of regularity and uniqueness of such solutions was clearly mentioned and left as an open problem. To our knowledge, the only existing result on the regularity of the DiPerna–Lions weak solutions is due to Bouchut, Golse and Pallard [10]. In this paper, the authors proved that the electromagnetic field belongs to $H^s_{\text{loc}}(\mathbb{R}^*_+ \times \mathbb{R}^3)$ with $s = 2/11$, under the assumption that the macroscopic kinetic energy density is square summable. The authors of [10] mentioned the natural issue of the optimality of their regularity result. In fact, their result is not optimal, since we prove here a slightly better regularity for the electromagnetic field. Indeed, under the same assumptions as [10, Theorem 2], we prove that the electromagnetic field belongs to $H^s_{\text{loc}}(\mathbb{R}^*_+ \times \mathbb{R}^3)$, with $s = 6/(13 + \sqrt{142}) > 3/13 > 2/11$. The regularity estimate of [10] results from a smoothing mechanism, called "nonresonant smoothing", which relies on the property that the euclidean norm of the velocity is less than unity for particle momentum staying in a compact set. This smoothing effect is reminiscent of the proof of existence and uniqueness of compactly supported $\mathscr{C}^1$ classical solutions for the relativistic Vlasov–Maxwell system performed in [21, 28]. Indeed, in these works, the boundedness of the electromagnetic field is controlled by the boundedness of the distribution function as long as the denominators $1 \pm v \cdot \xi/|\xi|$ (ξ is the Fourier dual variable of the space variable x) are bounded away from zero. This holds for momentum remaining in a compact set. The proof of the regularity result in [10] relies on two key ingredients: the first one is that some well-chosen combinations of the wave operator and the free-streaming operator lead to elliptic operators in time and space variables. This elliptic regularity occurs because, under the nonresonant smoothing condition, the intersection of characteristic manifolds of the wave and the free-streaming operators is empty. In this framework, the characteristic manifold of an operator is the set of value of time, space and their Fourier dual variables for which the symbol of the corresponding operator is null. The second ingredient is the kinetic formulation of Maxwell's equations: the latter are equivalently replaced by a scalar wave equation for a scalar potential depending on an extra variable, which is the particle momentum. Our proof is also based on the nonresonant smoothing mechanism but it follows the Fourier approach of [28], combined together with standard regularity results for the wave equation [30, 38]. As suggested by the seminal works [13, 25, 24] the regularity can be investigated by splitting momentum space in two regions combined together with interpolation inequalities. The first region is defined by $|p| \leq R$ whereas the second one is the complementary set $|p| > R$, with R chosen arbitrarily. Our improvement of the index of regularity comes from the contribution $|p| \leq R$ for which we obtain estimates that increase polynomially in R with a smaller exponent than in [10]. Using some L^q controls of the macroscopic kinetic energy density, the contribution $|p| > R$ leads to an estimate decreasing polynomially in R. This expresses that charge and current densities, and the electromagnetic field (through Maxwell's equations) created by the corresponding particles in this region are small as $R \to \infty$. Here, the polynomial exponent for the contribution $|p| > R$ is the same as in [10] because we assume the same condition on the control of the macroscopic kinetic energy density.

Finally, let us stress that regularity issue is important since it is closely related to the uniqueness one. Unfortunately such regularity result on the electromagnetic field is still insufficient to obtain uniqueness of weak solutions through a combination of Eulerian and Lagrangian formulations. Indeed, recently there has been an important development of existence and uniqueness theories of Lagrangian flows for transport equations with low regularity vector fields [14, 7, 1, 12, 8, 2, 26] (see, e.g. [4] for an excellent summary). Some of them have been successfully used to prove uniqueness of weak solutions for the Vlasov–Poisson system [3, 6, 33] and for a non-self-consistent (without feedback of particles on electromagnetic fields) Vlasov–Maxwell system [27].

Now, we state the main result of this paper, the following *a priori* regularity result on the electromagnetic field.

Theorem 1.1. *Consider initial conditions* (f_0, E_0, B_0) *such that* $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$, $f_0 \geq 0$ *a.e.,* E_0 *and* $B_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ *satisfy,*

$$\nabla \cdot B_0 = 0, \quad \nabla \cdot E_0 = \rho_0 = \int_{\mathbb{R}^3} f_0 \, dp.$$

Assume in addition that the energy bound $\mathcal{E}_0 := \mathcal{E}(t = 0) < +\infty$ *holds, with*

$$\mathcal{E}(t) := \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dp \sqrt{1 + |p|^2} f(t) + \int_{\mathbb{R}^3} (|E(t)|^2 + |B(t)|^2) \, dx.$$

Let (f, E, B) *be a weak solution of the relativistic Vlasov–Maxwell system* (1.1)–(1.4) *whose existence is proved in* [13] *and such that* $f \in L^\infty(0, \infty; L^1 \cap L^\infty(\mathbb{R}^6))$, E *and* $B \in L^\infty(0, \infty; L^2(\mathbb{R}^3))$, $\mathcal{E}(t) \leq \mathcal{E}_0$ *a.e.* $t \geq 0$, *and* $\|f(t)\|_{L^p(\mathbb{R}^6)} \leq \|f_0\|_{L^p(\mathbb{R}^6)}$ *a.e.* $t \geq 0$, *for* $p \in [1, +\infty]$. *If the macroscopic kinetic energy density satisfies,*

$$\int_{\mathbb{R}^3} \sqrt{1 + |p|^2} f \, dp \in L^q_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3), \quad \text{with } q \in]3/2, 2], \tag{1.8}$$

then the electromagnetic field has the following regularity:

$$E, B \in H^s_{\mathrm{loc}}(\mathbb{R}^*_+ \times \mathbb{R}^3), \quad \textit{with}$$

$$s < \frac{2q-3}{2q-3+\ell(q)}, \quad \ell(q) = \frac{19}{6} + q\left(-1 + \sqrt{1 - \frac{7}{6q} + \frac{41}{18q^2}}\right) \leq \frac{10}{3},$$

$$\textit{and} \quad q \in]3/2, 2].$$

Remark 1.2. Using averaging lemmas [25, 24, 13] and standard results for wave equation [30, 38], we obtain a first regularity result for the electromagnetic field (E, B). Tracing the constant R (the radius of the compact momentum ball B_R) in the proof of averaging lemmas (see, e.g. [15, Chap. 7]), we obtain,

$$\|\rho^{<R}\|_{H^{1/4}([0,T]\times\mathbb{R}^3)} \lesssim R^\sigma, \quad \|j^{<R}\|_{H^{1/4}([0,T]\times\mathbb{R}^3)} \lesssim R^\sigma, \quad \text{with } \sigma = 7/4, \tag{1.9}$$

where we define,

$$\rho^{<R} = \int_{|p|\leq R} f dp, \quad \text{and} \quad j^{<R} = \int_{|p|\leq R} v f dp.$$

Under condition (1.8) and using [10, Lemma 4] (see also Lemma 2.3), we obtain,

$$\|\rho^{>R}\|_{L^2([0,T]\times\mathbb{R}^3)} \lesssim R^{3-2q}, \quad \|j^{>R}\|_{L^2([0,T]\times\mathbb{R}^3)} \lesssim R^{3-2q}, \quad \text{with } q \in]3/2, 2], \tag{1.10}$$

where we define,

$$\rho^{>R} = \int_{|p|>R} f dp, \quad \text{and} \quad j^{>R} = \int_{|p|>R} v f dp.$$

Using the interpolation inequality $\|u\|_{H^{\alpha/4}} \leq \|u_1\|^{1-\alpha}_{L^2}\|u_2\|^{\alpha}_{H^{1/4}}$, with $u = u_1 + u_2$ (see, e.g. [5]), and estimates (1.9)–(1.10), we obtain,

$$\rho, j \in H^s_{\mathrm{loc}}(\mathbb{R}^*_+ \times \mathbb{R}^3), \quad \text{with } s = \frac{2q-3}{4(2q-3+\sigma)}. \tag{1.11}$$

Let us rewrite Maxwell's equations in terms of the scalar electrical potential ϕ and the magnetic vector potential A. The electromagnetic field (E, B) is then given by the usual formulas,

$$E = -\partial_t A - \nabla\phi, \quad B = \nabla \times A. \tag{1.12}$$

Using the wave-operator definition, $\Box \equiv \partial_t^2 - \Delta_x$, the electromagnetic potential (ϕ, A) satisfies the standard wave equations:

$$\begin{gathered} \Box\phi = \rho, \quad \Box A = j, \\ \phi_{|t=0} = \phi_0, \quad A_{|t=0} = A_0, \\ \partial_t\phi_{|t=0} = \partial_t\phi_0, \quad \partial_t A_{|t=0} = \partial_t A_0. \end{gathered} \tag{1.13}$$

For initial conditions, we assume the following regularity: ϕ_0, $A_0 \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $\partial_t\phi_0, \partial_t A_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$. In addition, initial conditions must satisfy the Lorentz gauge,

$$\partial_t\phi_0 + \nabla \cdot A_0 = 0. \tag{1.14}$$

The electromagnetic potential (ϕ, A) then satisfies the Lorentz gauge condition at any time, namely,

$$\partial_t\phi + \nabla \cdot A = 0. \tag{1.15}$$

Indeed, setting $g := \partial_t\phi + \nabla \cdot A$, from the charge conservation law (1.7) and the wave equations (1.13) we obtain $\Box g = \partial_t\rho + \nabla \cdot j = 0$; from the Lorentz gauge (1.14) we have $g_{|t=0} = 0$; from the wave equation (1.13) for ϕ, the electromagnetic field definition (1.12) and Maxwell–Gauss law (1.3), we obtain $\partial_t g_{|t=0} = 0$; hence $g = 0$, i.e. (1.15).

To construct, with the desired regularity, the initial conditions ϕ_0, A_0, $\partial_t\phi_0$, and $\partial_t A_0$ of the wave equations (1.13) from the initial conditions E_0 and B_0 of Theorem 1.1, we can proceed as follows. Without loss of generality, we can choose $\nabla \cdot A_0 = 0$ in (1.14) and thus we obtain the initial condition $\partial_t\phi_0 = 0$. From $\nabla \cdot A_0 = 0$ and $\nabla \times A_0 = B_0$, where B_0 is given by assumptions of Theorem 1.1, we can determine A_0. Indeed, $\nabla \cdot A_0 = 0$ implies that there exists a vector Ψ_0 such that $A_0 = \nabla \times \Psi_0$ and for which we can choose the gauge condition $\nabla \cdot \Psi_0 = 0$. Therefore, we have $-\Delta\Psi_0 = B_0$, with $B_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, and the initial condition A_0 is then given by $A_0 = -\nabla \times \Delta^{-1} B_0 \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$. Since Eq. (1.12) is also satisfied at initial time, we can choose the initial condition $\partial_t A_0 = 0$, and we obtain $E_0 = -\nabla\phi_0$. Maxwell–Gauss law (1.3) and the regularity assumption $E_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, imply that ϕ_0 satisfies $-\Delta\phi_0 = \rho_0$, with $\rho_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Therefore, the initial condition ϕ_0 is given by $\phi_0 = -\Delta^{-1}\rho_0 \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$.

Now, using standard regularity results for the wave equation (see, e.g. [30, Chap. 3] or [38, Chap. 4]) and the regularity of charge and current densities (1.11), we obtain ϕ, $A \in L^2 \cap L^\infty \cap \mathscr{C}(0, T; H^{s+1}_{\mathrm{loc}}(\mathbb{R}^3))$ and $\partial_t\phi$, $\partial_t A \in L^2 \cap L^\infty \cap \mathscr{C}(0, T; H^s_{\mathrm{loc}}(\mathbb{R}^3))$. These regularity properties and (1.12) imply,

$$E, B \in L^2 \cap L^\infty \cap \mathscr{C}(0, T; H^s_{\mathrm{loc}}(\mathbb{R}^3)),$$

with s given by (1.11).

Remark 1.3. Under the same assumptions as Theorem 1.1, the authors of [10] obtain,

$$E, B \in H^s_{\mathrm{loc}}(\mathbb{R}^*_+ \times \mathbb{R}^3), \quad \text{with } s < \frac{4q-6}{4q+3}, \quad \text{and} \quad q \in]3/2, 2].$$

2. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. For this purpose, we first recall in Sec. 2.1 an alternative formulation of Maxwell's system, which appears in [21, 28]. In Sec. 2.2, we estimate the regularity of terms coming from initial conditions.

In Sec. 2.3, we establish estimates in H^1 norm for the contribution from low velocity particles, the momentum of which is such that $|p| \leq R$. In Sec. 2.4, we control in L^2 and H^1 norms the contribution from high velocity particles, the momentum of which is such that $|p| > R$. Finally in Sec. 2.5, we gather all estimates of previous sections and complete the proof to derive the regularity result of Theorem 1.1.

2.1. *Reformulation of Maxwell's equations*

The formulation of Maxwell's equations in [21, 28], consists in writing wave equations for (E, B), where the source terms are rewritten by using the Vlasov equation to deal with space-time derivatives of charge and current densities. Indeed, combining Maxwell's equations (1.2) and (1.3), using Vlasov equation (1.1) for re-expressing the term $\partial_t f$, and using also the definition of charge and current densities (1.4), we obtain componentwise, for $k \in \{1, 2, 3\}$,

$$\partial_t^2 E_k - \Delta E_k = \int_{\mathbb{R}^3} (v_k v_l \partial_{x_l} f - \partial_{x_k} f + v_k [E_l + \varepsilon_{lij} v_i B_j] \partial_{p_l} f)\, dp, \tag{2.1}$$

$$\partial_t^2 B_k - \Delta B_k = \varepsilon_{klm} \int_{\mathbb{R}^3} v_m \partial_{x_l} f\, dp, \tag{2.2}$$

where ε_{ijk} is the antisymmetric Levi-Civita symbol. Here, we use the convention that an index variable appearing twice in a single term implies the summation of that term over all the values of the index. To write a single vector wave equation as in [28], we introduce the electromagnetic field $\Phi : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}^6$, defined by $\Phi = (E^T, B^T)^T$. Then Maxwell's equations (2.1) and (2.2) become,

$$\partial_t^2 \Phi_k - \Delta \Phi_k = J_k := \int_{\mathbb{R}^3} (M_{kl} \partial_{x_l} f + N_{klm} \Phi_m \partial_{p_l} f)\, dp, \tag{2.3}$$

where $M = M(v)$ is a 6-by-3 real matrix and $N = N(v)$ is a 6-by-3-by-6 real tensor of rank 3. These tensors depend only on the velocity variable v and are defined as follows. We first introduce the antisymmetric matrix $Q = Q(v)$, which is associated to the cross product with the vector v and defined by,

$$Q(v) = \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix}. \tag{2.4}$$

For any three-dimensional vector ω, we have $Q(v)\omega = \omega \times v$ and $Q^T(v)\omega = -Q(v)\omega = v \times \omega$. We also introduce the 3-by-6 real matrix $\alpha = \alpha(v)$ defined by,

$$\alpha(v) = (I_3, Q^T(v)), \tag{2.5}$$

where I_3 is the 3-by-3 identity matrix. We then obtain,

$$M(v) = \begin{pmatrix} v \otimes v - I_3 \\ Q(v) \end{pmatrix}, \quad \text{and} \quad N_{klm}(v) = \begin{cases} v_k \alpha_{lm}(v) & \text{if } k \leq 3, \\ 0 & \text{if } k > 3. \end{cases} \tag{2.6}$$

Using the fundamental solution of the wave operator $\Box \equiv \partial_t^2 - \Delta$ (see, e.g. [38]), we obtain the following integral representation for the electromagnetic field Φ: for

$i \in \{1, \dots, 6\}$,

$$\Phi_i(t,x) = \int_{\mathbb{R}^3} d\xi \, e^{\mathrm{i}x\cdot\xi} \left(\cos(|\xi|t)\widehat{\Phi}_{0i}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\widehat{\Phi}_{1i}(\xi) - \int_0^t \frac{\sin(|\xi|(t-\sigma))}{|\xi|}\widehat{J}_i(\sigma,\xi)d\sigma \right), \tag{2.7}$$

where

$$\widehat{\Phi}_{0i} = \mathcal{F}_x(\Phi_{0i}) = \mathcal{F}_x(\Phi_i(0,\cdot)), \quad \widehat{\Phi}_{1i} = \mathcal{F}_x(\Phi_{1i}) = \mathcal{F}_x(\partial_t\Phi_i(0,\cdot)), \quad \widehat{J}_i = \mathcal{F}_x(J_i).$$

Here, we use the notation $\widehat{J} \equiv \mathcal{F}_x(J)$ where $\mathcal{F}_x$ denotes the Fourier transform with respect to the space variable x, and is defined by,

$$\hat{g}(\xi) = \mathcal{F}_x(g)(\xi) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} g(x)e^{-\mathrm{i}x\cdot\xi}\,dx, \quad \text{and}$$

$$g(x) = \mathcal{F}_\xi^{-1}(\hat{g})(x) = \int_{\mathbb{R}^3} \hat{g}(\xi)e^{\mathrm{i}x\cdot\xi}\,d\xi.$$

Since we want to obtain local estimates in space and time, we multiply (2.7) by a test function $\eta = \eta(t,x) \in \mathcal{D}(\mathbb{R}_+^* \times \mathbb{R}^3)$ and we set $\Phi_\eta := \Phi\eta$. Now, we split Φ_η into three parts,

$$\Phi_\eta = \Phi_\eta^0 + \Phi_\eta^{<R} + \Phi_\eta^{>R},$$

where Φ_η^0, $\Phi_\eta^{<R}$, and $\Phi_\eta^{>R}$ denote the contributions of the initial conditions, of the low velocity particles and the contribution of the high velocity particles, respectively. We define the term Φ_η^0 by,

$$\Phi_{\eta i}^0(t,x) = \int_{\mathbb{R}^3} d\xi \, e^{\mathrm{i}\xi\cdot x}\eta(t,x) \left(\cos(|\xi|t)\widehat{\Phi}_{0i}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\widehat{\Phi}_{1i}(\xi) \right). \tag{2.8}$$

We define the contribution $\Phi_\eta^{<R}$ by,

$$\Phi_{\eta i}^{<R}(t,x) = -\int_{\mathbb{R}^3} d\xi \, e^{\mathrm{i}\xi\cdot x}\eta(t,x) \int_0^t \frac{\sin(|\xi|(t-\sigma))}{|\xi|}\widehat{J}_i^{<R}(\sigma,\xi)d\sigma, \tag{2.9}$$

with

$$\widehat{J}_i^{<R}(\sigma,\xi) = \int_{|p|\le R} (\mathrm{i}M_{ij}(v)\xi_j \hat{f}(\sigma,\xi,p) + N_{ijk}(v)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \partial_{p_j}\hat{f}(\sigma,\xi,p))\,dp, \tag{2.10}$$

where we have used the Fourier transform of the right-hand side of (2.3). The contribution $\Phi_\eta^{>R}$, arising from high velocity particles, is defined analogously. In the sequel, we shall estimate each of the three terms Φ_η^0, $\Phi_\eta^{<R}$ and $\Phi_\eta^{>R}$.

2.2. *Contribution from initial conditions*

Here, we give a local estimate of the term Φ^0_η in H^1 norm. From assumptions of Theorem 1.1, we have $\Phi_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $\Phi_1 = 0$. Then, first-order derivatives in space and time of (2.8) are,

$$\Phi^0_{\eta i}(t,x) = \int_{\mathbb{R}^3} d\xi\, e^{\mathrm{i}\xi\cdot x} K_\eta(t,x,\xi)\widehat{\Phi}_{0i}, \tag{2.11}$$

$$\partial_t\Phi^0_{\eta i}(t,x) = \int_{\mathbb{R}^3} d\xi\, e^{\mathrm{i}\xi\cdot x} K^t_\eta(t,x,\xi)\widehat{\Phi}_{0i}, \tag{2.12}$$

$$\partial_{x_j}\Phi^0_{\eta i}(t,x) = \int_{\mathbb{R}^3} d\xi\, e^{\mathrm{i}\xi\cdot x} K^{x_j}_\eta(t,x,\xi)\widehat{\Phi}_{0i}. \tag{2.13}$$

The symbols K_η, K^t_η and $K^{x_j}_\eta$ are compactly supported in space and time. Denoting by S^m the class of standard symbols of order m (see, e.g. [40, Sec. 1.3, Chap. VI]), we have $K_\eta \in S^0$, $K^t_\eta \in S^1$ and $K^{x_j}_\eta \in S^1$, uniformly with respect to time in a compact set. Indeed, we have the following estimates:

$$K_\eta(t,x,\xi) = \eta(t,x)\cos(|\xi|t) \le C(\|\eta\|_{L^\infty(\mathbb{R}^*_+\times\mathbb{R}^3)}),$$

$$K^t_\eta(t,x,\xi) = \partial_t\eta(t,x)\cos(|\xi|t) - |\xi|\eta(t,x)\sin(|\xi|t) \le C(\|\eta\|_{W^{1,\infty}(\mathbb{R}^*_+\times\mathbb{R}^3)})|\xi|,$$

$$K^{x_j}_\eta(t,x,\xi) = \partial_{x_j}\eta(t,x)\cos(|\xi|t) - t\xi_j\eta(t,x)\sin(|\xi|t) \le C(T,\|\eta\|_{W^{1,\infty}(\mathbb{R}^*_+\times\mathbb{R}^3)})|\xi|.$$

Therefore, from standard results on pseudo-differential operators (see, e.g. [40, Proposition 5, Sec. 5.2, Chap. VI]) and the regularity assumption $\Phi_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$, terms (2.11)–(2.13) are bounded in $L^2(\mathbb{R}^*_+ \times \mathbb{R}^3)$ and we obtain,

$$\|\Phi^0\|_{H^1_{\mathrm{loc}}(\mathbb{R}^*_+\times\mathbb{R}^3)} \le C(T)\|\Phi_0\|_{H^1_{\mathrm{loc}}(\mathbb{R}^3)}. \tag{2.14}$$

2.3. *Contribution from low velocity particles*

In this section, we give a local estimate of the term $\Phi^{<R}$ in H^1 norm. This result is summarized in Proposition 2.1, the proof of which is divided in several technical steps. As in [28], the first step (see Sec. 2.3.1) consists in rewriting the term $\Phi^{<R}$ to take benefit of the nonresonant smoothing effect, which leads to a gain of one order of derivative or regularity in space variables. In a second step, we establish the technical Lemma 2.2 that we use with standard regularity results on pseudo-differential and Fourier integral operators (see, e.g. [40]) to estimate, through several sections, each piece of $\Phi^{<R}$ in H^1 norm. Finally, gathering all these estimates we obtain

Proposition 2.1. *There exists a constant* C, *depending on* $\|\Phi_0\|_{L^2(\mathbb{R}^3)}$ *and* $\|f_0\|_{L^2\cap L^\infty(\mathbb{R}^6)}$, *such that,*

$$\|\Phi^{<R}\|_{H^1_{\mathrm{loc}}(R^*_+\times\mathbb{R}^3)} \le CR^{10/3}. \tag{2.15}$$

2.3.1. *The well-suited form of* $\Phi^{<R}$

Let us first start by rewriting the term $\Phi^{<R}$. Using Euler formula for the sinus function, from (2.9)–(2.10), we obtain,

$$\Phi^{<R}_{\eta i} = I^+_i + I^-_i,$$

where

$$\begin{aligned} I^\pm_i = I^\pm_{1i} + I^\pm_{2i} &= \mp\frac{1}{2}\int_0^t d\sigma \int_{\mathbb{R}^3} d\xi \int_{|p|\le R} dp\, e^{\mathrm{i}[\pm|\xi|(t-\sigma)+x\cdot\xi]}\eta(t,x)M_{ij}(v)\frac{\xi_j}{|\xi|}\hat{f}(\sigma,\xi,p) \\ &\quad \pm\frac{\mathrm{i}}{2}\int_0^t d\sigma \int_{\mathbb{R}^3} d\xi \int_{|p|\le R} dp\, e^{\mathrm{i}[\pm|\xi|(t-\sigma)+x\cdot\xi]}\eta(t,x) \\ &\quad \times \frac{1}{|\xi|}N_{ijk}(v)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \partial_{p_j}\hat{f}(\sigma,\xi,p). \end{aligned}$$

As it was done in [28] for terms $I^\pm_{1i}$, we take benefit of the nonresonant smoothing mechanism, that consists in obtaining an additional one order of regularity (or space derivative) represented by the factor $1/|\xi|$ when integrating by parts in time terms $I^\pm_{1i}$. This smoothness effect holds provided that $|1 \pm v\cdot\xi/|\xi||^{-1} < +\infty$, which is the case for momentum p staying in a compact set. Indeed, using an integration by parts in time, we have,

$$\int_0^t d\sigma e^{\mp\mathrm{i}|\xi|\sigma}\hat{f}(\sigma,\xi,p) = \pm\frac{\mathrm{i}}{|\xi|}\left(\left[e^{\mp\mathrm{i}\sigma|\xi|}\hat{f}\right]_{\sigma=0}^{\sigma=t} - \int_0^t d\sigma e^{\mp\mathrm{i}\sigma|\xi|}\partial_\sigma\hat{f}(\sigma,\xi,p)\right). \tag{2.16}$$

Using the Vlasov equation (1.1) written in Fourier variable ξ, i.e.,

$$\partial_\sigma\hat{f} + \mathrm{i}v\cdot\xi\hat{f} + \alpha_{ij}\widehat{\Phi}_j \underset{\xi}{*} \partial_{p_i}\hat{f} = 0, \tag{2.17}$$

to replace the term $\partial_\sigma\hat{f}$ in (2.16), we obtain,

$$\int_0^t d\sigma e^{\mp\mathrm{i}|\xi|\sigma}\hat{f} = \pm\frac{\mathrm{i}}{|\xi|}\frac{1}{D^\pm}\left(\left[e^{\mp\mathrm{i}\sigma|\xi|}\hat{f}\right]_{\sigma=0}^{\sigma=t} - \int_0^t d\sigma e^{\mp\mathrm{i}\sigma|\xi|}\alpha_{ij}\widehat{\Phi}_j \underset{\xi}{*} \partial_{p_i}\hat{f}\right),$$

where we set,

$$\begin{aligned} D^\pm &:= 1 \pm \frac{v\cdot\xi}{|\xi|} = 1 \pm v\cdot\omega, \quad \text{with } \omega = \omega(\xi) := \frac{\xi}{|\xi|}, \quad \text{and} \\ v &= v(p) := \frac{p}{\sqrt{1+|p|^2}}. \end{aligned} \tag{2.18}$$

Substituting (2.16) in terms $I^\pm_{1i}$, we obtain,

$$\begin{aligned} I^\pm_i &= -\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi \int_{|p|\le R} dp\, e^{\mathrm{i}[\pm|\xi|t+x\cdot\xi]}\eta(t,x)M_{ij}(v)\frac{\xi_j}{|\xi|^2}\frac{1}{D^\pm}\left[e^{\mp\mathrm{i}\sigma|\xi|}\hat{f}\right]_{\sigma=0}^{\sigma=t} \\ &\quad + \frac{\mathrm{i}}{2}\int_0^t d\sigma \int_{\mathbb{R}^3} d\xi \int_{|p|\le R} dp\, e^{\mathrm{i}[\pm|\xi|(t-\sigma)+x\cdot\xi]}\eta(t,x) \\ &\quad \times \frac{1}{|\xi|}\left(M_{ij}(v)\frac{\xi_j}{|\xi|}\frac{1}{D^\pm}\alpha_{lk}(v) \pm N_{ilk}(v)\right)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \partial_{p_l}\hat{f}(\sigma,\xi,p). \end{aligned} \tag{2.19}$$

Integrating by parts in momentum variable p the second integral of the right-hand side of (2.19), we obtain the convenient form of terms $I_i^\pm$ given by,

$$
\begin{aligned}
I_i^\pm &= I_{1i}^\pm + I_{2i}^\pm + I_{3i}^\pm + I_{4i}^\pm \\
&= -\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi \int_{|p|\le R} dp\, e^{\mathrm{i}x\cdot\xi}\eta(t,x)\frac{1}{|\xi|}\frac{1}{D^\pm}M_{ij}(v)\omega_j\hat{f}(t,\xi,p) \\
&\quad + \frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi \int_{|p|\le R} dp\, e^{\mathrm{i}[\pm|\xi|t + x\cdot\xi]}\eta(t,x)\frac{1}{|\xi|}\frac{1}{D^\pm}M_{ij}(v)\omega_j\hat{f}_0(\xi,p) \\
&\quad + \frac{\mathrm{i}}{2}\int_0^t d\sigma \int_{\mathbb{R}^3} d\xi \int_{|p|=R} dp\, e^{\mathrm{i}[\pm|\xi|(t-\sigma)+x\cdot\xi]}\eta(t,x) \\
&\quad \times \frac{1}{|\xi|}\left(\frac{1}{D^\pm}M_{ij}(v)\omega_j\alpha_{lk}(v) \pm N_{ilk}(v)\right)\nu_l(p)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \hat{f}(\sigma,\xi,p) \\
&\quad - \frac{\mathrm{i}}{2}\int_0^t d\sigma \int_{\mathbb{R}^3} d\xi \int_{|p|\le R} dp\, e^{\mathrm{i}[\pm|\xi|(t-\sigma)+x\cdot\xi]}\eta(t,x) \\
&\quad \times \frac{1}{|\xi|}\partial_{p_l}\left(\frac{1}{D^\pm}M_{ij}(v)\omega_j\alpha_{lk}(v) \pm N_{ilk}(v)\right)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \hat{f}(\sigma,\xi,p),
\end{aligned}
$$

where $\nu(p) := p/|p|$ is the normal unit vector to the momentum sphere of radius $|p|$. Before estimating terms $I_{ji}^\pm$, we establish

Lemma 2.2. *Let κ be a pure numerical constant and ω be any three-dimensional unit vector. Then the following estimates hold,*

$$|v|, |\alpha_{ij}|, |N_{ijk}| \le 1, \quad |M_{ij}| \le 2, \tag{2.20}$$

$$|\nabla_p v|, |\nabla_p \alpha_{ij}|, |\nabla_p N_{ijk}|, |\nabla_p M_{ij}| \le \kappa/\sqrt{1+|p|^2}, \tag{2.21}$$

$$0 \le (2(1+|p|^2))^{-1} \le D^\pm \le 2, \tag{2.22}$$

$$|\omega - (v\cdot\omega)v|, \quad |v\times\omega| \le \sqrt{2D^\pm}, \tag{2.23}$$

$$\int_{|p|\le R}\frac{dp}{D^\pm} \le \kappa R^4, \quad \int_{|p|=R}\frac{dp}{D^\pm} \le \kappa R^3, \quad \int_{|p|\le R}\frac{dp}{(\sqrt{1+|p|^2}D^\pm)^2} \le \kappa R^{11/3}. \tag{2.24}$$

Proof. From definitions (2.4)–(2.6) and (2.18), estimates (2.20)–(2.21) and the upper bound of (2.22) are straightforward. For the lower bound of (2.22), we have,

$$
\begin{aligned}
D^\pm \ge 1 - |v| \ge 1 - \frac{|p|}{\sqrt{1+|p|^2}} &\ge \frac{\sqrt{1+|p|^2}-|p|}{\sqrt{1+|p|^2}} \\
&\ge \frac{1}{\sqrt{1+|p|^2}(\sqrt{1+|p|^2}+|p|)} \ge \frac{1}{2(1+|p|^2)}.
\end{aligned}
$$

Let us show estimates (2.23). Using definitions (2.18), we first have,

$$\begin{aligned} |\omega-(v\cdot\omega)v| &= (1-2(v\cdot\omega)^2+(v\cdot\omega)^2|v|^2)^{1/2}\\ &\le (1-(v\cdot\omega)^2)^{1/2}=\sqrt{D^+D^-}\le\sqrt{2D^+}, \qquad (2.25)\\ |v\pm\omega| &= (|v|^2\pm 2v\cdot\omega+1)^{1/2}\le\sqrt{2D^\pm},\\ |v\times\omega| &= |(v\pm\omega)\times\omega|\le|v\pm\omega|\le\sqrt{2D^\pm}. \end{aligned}$$

To end the proof of (2.23), we set $\omega = -\tilde{\omega}$ in $|\omega-(v\cdot\omega)v|$, which leads to $|\tilde{\omega}-(v\cdot\tilde{\omega})v|$ and we apply the series of inequalities (2.25). Let us prove the first estimate of (2.24). We only deal with the case $D := D^+$, since the case D^- follows the same proof. Using (2.18), we obtain,

$$D^{-1}=\frac{\sqrt{1+|p|^2}(\sqrt{1+|p|^2}-\omega\cdot p)}{1+|p|^2-(\omega\cdot p)^2}. \tag{2.26}$$

We set the angle between the vector p and ω to $\theta+\pi$. Then (2.26) becomes,

$$D^{-1}=\frac{\sqrt{1+|p|^2}(\sqrt{1+|p|^2}+|p|\cos\theta)}{1+|p|^2\sin^2\theta}.$$

For θ small enough, we claim that we have,

$$D^{-1}\le 4\frac{1+|p|^2}{1+|p|^2\theta^2}. \tag{2.27}$$

Indeed, for θ small enough ($|\theta|<\pi/6$), we have $\theta^2/2\le\sin^2\theta\le\theta^2$. It follows that $1/(1+|p|^2\theta^2)\le 1/(1+|p|^2\sin^2\theta)\le 1/(1+|p|^2\theta^2/2)$ and we obtain,

$$D^{-1}\le 2\frac{1+|p|^2}{1+|p|^2\theta^2/2}\le 4\frac{1+|p|^2}{2+|p|^2\theta^2}\le 4\frac{1+|p|^2}{1+|p|^2\theta^2},$$

which proves (2.27). Let $0<\theta_0<1$, and assume $|\theta|>\theta_0$. Using (2.27), we obtain,

$$D^{-1}\le 4\frac{1+|p|^2}{1+|p|^2\theta^2}\le 4\frac{1+|p|^2}{1+|p|^2\theta_0^2}\le\frac{4}{\theta_0^2}. \tag{2.28}$$

Inequality (2.28) holds because the function $\mathbb{R}^+\ni t\mapsto(1+t)/(1+\theta t)\in\mathbb{R}^+$ is nonincreasing if $\theta>1$ (with a maximum value 1) and is nondecreasing if $\theta<1$ (with a maximum value $1/\theta$). Using (2.28) with θ_0 small enough, estimate (2.22), and spherical coordinates where the zenith direction is taken in the opposite direction of ω, we obtain,

$$\int_{|p|\le R}\frac{dp}{D}\le\int_{|p|\le R,|\theta|>\theta_0}\frac{dp}{D}+\int_{|p|\le R,|\theta|\le\theta_0}\frac{dp}{D}$$

$$
\begin{aligned}
&\le \frac{4}{\theta_0^2}\int_{|p|\le R,|\theta|>\theta_0} dp + 4R^2\int_{|p|\le R,|\theta|\le\theta_0} dp \\
&\le \frac{16\pi R^3}{3\theta_0^2} + 4R^2\int_0^{2\pi} d\varphi \int_0^{\theta_0} \sin\theta d\theta \int_0^R |p|^2 d|p| \\
&\le \frac{16\pi R^3}{3\theta_0^2} + \frac{8\pi}{3}R^5(1-\cos\theta_0) \\
&\le \frac{16\pi R^3}{3\theta_0^2} + \frac{4\pi}{3}R^5\theta_0^2 \le \frac{16\pi}{3}\left(\frac{R^3}{\theta_0^2} + R^5\theta_0^2\right) \\
&\le \kappa R^4.
\end{aligned}
$$

The last inequality is obtained by taking the best θ_0, which is given by $\theta_0 = 1/\sqrt{R}$. We observe that θ_0 is very small as R is large. For the second estimate of (2.24), we obtain,

$$
\begin{aligned}
\int_{|p|=R}\frac{dp}{D} &\le \int_{|p|=R,|\theta|>\theta_0}\frac{dp}{D} + \int_{|p|=R,|\theta|\le\theta_0}\frac{dp}{D} \\
&\le \frac{4}{\theta_0^2}\int_{|p|\le R,|\theta|>\theta_0} dp + 4R^2\int_{|p|=R,|\theta|\le\theta_0} dp \\
&\le \frac{16\pi R^2}{\theta_0^2} + 4R^2\int_0^{2\pi} d\varphi \int_0^{\theta_0} R^2 \sin\theta d\theta \\
&\le \frac{16\pi R^2}{\theta_0^2} + 8\pi R^4(1-\cos\theta_0) \\
&\le \frac{16\pi R^2}{\theta_0^2} + 8\pi R^4\theta_0^2 \le 16\pi\left(\frac{R^2}{\theta_0^2} + R^4\theta_0^2\right) \\
&\le \kappa R^3.
\end{aligned}
$$

The last inequality is obtained by taking the best θ_0, which is also given by $\theta_0 = 1/\sqrt{R}$. Finally, for last estimate of (2.24), we obtain,

$$
\begin{aligned}
\int_{|p|\le R}\frac{dp}{(\sqrt{1+|p|^2}D)^2} &\le \int_{|p|\le R,|\theta|>\theta_0}\frac{dp}{(\sqrt{1+|p|^2}D)^2} + \int_{|p|\le R,|\theta|\le\theta_0}\frac{dp}{(\sqrt{1+|p|^2}D)^2} \\
&\le \frac{16}{\theta_0^4}\int_{|p|\le R,|\theta|>\theta_0}\frac{dp}{1+|p|^2} + 4R^2\int_{|p|\le R,|\theta|\le\theta_0} dp \\
&\le 64\pi\left(\frac{R}{\theta_0^4} + R^5\theta_0^2\right) \\
&\le \kappa R^{11/3}.
\end{aligned}
$$

The last inequality is obtained by taking the best θ_0, which is given by $\theta_0 = R^{-2/3}$. □

2.3.2. *A priori estimate for* $I_{4i}^{\pm}$

We start by estimating $\|I_{4i}^{\pm}\|_{L^2(0,T;H^1(\mathbb{R}^3))}$ and next $\|\partial_t I_{4i}^{\pm}\|_{L^2(0,T;L^2(\mathbb{R}^3))}$. Setting,

$$\varphi_{\pm}(t,\sigma,x,\xi) = \pm|\xi|(t-\sigma) + x\cdot\xi, \tag{2.29}$$

and,

$$\hat{g}_i(\sigma,\xi) = \int_{|p|\le R} dp\,\frac{\partial}{\partial p_l}\left(\frac{1}{D^{\pm}}M_{ij}(v)\omega_j\alpha_{lk}(v) \pm N_{ilk}(v)\right)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \hat{f}(\sigma,\xi,p), \tag{2.30}$$

we have,

$$\|I_{4i}^{\pm}(t)\|_{H^1(\mathbb{R}^3)} \le \frac{1}{2}\int_0^t d\sigma\left\|\int_{\mathbb{R}^3} d\xi\frac{e^{\mathrm{i}\varphi_{\pm}(t,\sigma,x,\xi)}}{|\xi|}\eta(t,x)\hat{g}_i(\sigma,\xi)\right\|_{H^1(\mathbb{R}^3)}. \tag{2.31}$$

We observe that phases $\varphi_{\pm}$ are real-valued smooth functions in their arguments (x,ξ), homogeneous of degree 1 in ξ, and such that $\det(\nabla_{x,\xi}\varphi_{\pm}) = 1 \neq 0$. Moreover, the symbol $\eta/|\xi|$ of the Fourier integral operator in (2.31) belongs to S^{-1} and is compactly supported in space variables. Using standard results on Fourier integral operators (see, e.g. [40, Secs. 3 & 6.17, Chap. IX]), the Plancherel theorem, and the Cauchy–Schwarz inequality for p-integration in (2.30), we obtain from (2.31),

$$\begin{aligned}\|I_{4i}^{\pm}(t)\|_{H^1(\mathbb{R}^3)}^2 &\le C(T,\eta)\int_0^t d\sigma\|g_i(\sigma)\|_{L^2(\mathbb{R}^3)}^2 \le C(T,\eta)\int_0^t d\sigma\|\hat{g}_i(\sigma)\|_{L^2(\mathbb{R}^3)}^2\\ &\le C(T,\eta)\int_0^t d\sigma\|\Phi_k f(\sigma)\|_{L^2(\mathbb{R}^6)}^2 \sup_{\xi\in\mathbb{R}^3}\int_{|p|\le R} dp\,(|\partial_{p_l}N_{ilk}|\\ &\quad + |M_{ij}\omega_j\partial_{p_l}(1/D^{\pm})\alpha_{lk}| + |\partial_{p_l}M_{ij}\omega_j\alpha_{lk}/D^{\pm}|\\ &\quad + |M_{ij}\omega_j\partial_{p_l}\alpha_{lk}/D^{\pm}|)^2.\end{aligned} \tag{2.32}$$

We observe that $\partial_{p_l}\alpha_{lk} = 0$, for all $k\in\{1,\ldots,6\}$, and

$$M\omega = \Lambda, \quad (M\omega\nabla_p(1/D^{\pm})\alpha)_{ik} = \pm\frac{1}{\sqrt{1+|p|^2}}\frac{1}{(D^{\pm})^2}\Lambda_i\tilde{\Lambda}_k, \tag{2.33}$$

with

$$\Lambda = \begin{pmatrix}\omega-(v\cdot\omega)v\\ v\times\omega\end{pmatrix}, \quad \text{and} \quad \tilde{\Lambda} = \begin{pmatrix}\omega-(v\cdot\omega)v\\ \omega\times v\end{pmatrix}. \tag{2.34}$$

Using Lemma 2.2, from (2.32)–(2.34), we obtain,

$$\begin{aligned}\|I_{4i}^{\pm}\|_{L^2(0,T;H^1(\mathbb{R}^3))}^2 &\le C(T,\eta)\|\Phi\|_{L^2(0,T;L^2(\mathbb{R}^3))}^2\|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^6))}^2 R^3\\ &\quad\times\sup_{\xi\in\mathbb{R}^3}\int_{|p|\le R}\frac{dp}{(\sqrt{1+|p|^2}D^{\pm})^2}\\ &\le C(T,\eta,\|\Phi_0\|_{L^2(\mathbb{R}^3)},\|f_0\|_{L^\infty(\mathbb{R}^6)})R^{20/3}.\end{aligned} \tag{2.35}$$

Let us now deal with terms $\partial_t I^{\pm}_{4i}$. Differentiating terms $I^{\pm}_{4i}$ with respect to time, we obtain,

$$\partial_t I^{\pm}_{4i} = I^{\pm}_{41i} + I^{\pm}_{42i} = -\frac{\mathrm{i}}{2}\int_0^t d\sigma \int_{\mathbb{R}^3} d\xi\, e^{\mathrm{i}\varphi_{\pm}(t,\sigma,x,\xi)}(\partial_t\eta(t,x)/|\xi| \pm \mathrm{i}\eta(t,x))\hat{g}(\sigma,\xi)$$
$$- \frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi \frac{e^{\mathrm{i}x\cdot\xi}}{|\xi|}\eta(t,x)\hat{g}_i(t,\xi).$$

Terms $I^{\pm}_{41i}$ define Fourier integral operators with symbols in S^0, while terms $I^{\pm}_{42i}$ define pseudo-differential operators with symbols in S^{-1}. Following the same analysis that we have done for the estimate of $\|I^{\pm}_{4i}\|_{L^2(0,T;H^1(\mathbb{R}^3))}$, and using standard results on Fourier integral operators (see, e.g. [40, Sec. 3, Chap. IX]) and pseudo-differential operators (see, e.g. [40, Proposition 5, Sec.5.2, Chap. VI]) we then obtain,

$$\|\partial_t I^{\pm}_{4i}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \le C(T,\eta,\|\Phi_0\|_{L^2(\mathbb{R}^3)},\|f_0\|_{L^\infty(\mathbb{R}^6)})R^{11/3}.$$

This estimate and (2.35) lead to,

$$\|I^{\pm}_{4i}\|_{H^1([0,T]\times\mathbb{R}^3)} \le C(T,\eta,\|\Phi_0\|_{L^2(\mathbb{R}^3)},\|f_0\|_{L^\infty(\mathbb{R}^6)})R^{11/3}. \tag{2.36}$$

2.3.3. *A priori estimate for* $I^{\pm}_{3i}$

Here, we give an estimate for $\|I^{\pm}_{3i}\|_{L^2(0,T;H^1(\mathbb{R}^3))}$ and for $\|\partial_t I^{\pm}_{3i}\|_{L^2(0,T;L^2(\mathbb{R}^3))}$. Using (2.29) and defining,

$$\hat{g}_i(\sigma,\xi) = \int_{|p|=R} dp\left(\frac{1}{D^{\pm}}M_{ij}(v)\omega_j\alpha_{lk}(v) \pm N_{ilk}(v)\right)\nu_l(p)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \hat{f}(\sigma,\xi,p), \tag{2.37}$$

we have,

$$\|I^{\pm}_{3i}(t)\|_{H^1(\mathbb{R}^3)} \le \frac{1}{2}\int_0^t d\sigma \left\|\int_{\mathbb{R}^3} d\xi \frac{e^{\mathrm{i}\varphi_{\pm}(t,\sigma,x,\xi)}}{|\xi|}\eta(t,x)\hat{g}_i(\sigma,\xi)\right\|_{H^1(\mathbb{R}^3)}. \tag{2.38}$$

Let $\mathbb{S}^2_R$ be the two-dimensional momentum sphere, defined by the equation $|p| = R$. Using standard results on Fourier integral operators, the Plancherel theorem, and using the Cauchy–Schwarz inequality for p-integration on the sphere $\mathbb{S}^2_R$ in (2.37), we obtain from (2.38),

$$\|I^{\pm}_{3i}(t)\|^2_{H^1(\mathbb{R}^3)} \le C(T,\eta)\int_0^t d\sigma\|g_i(\sigma)\|^2_{L^2(\mathbb{R}^3)} \le C(T,\eta)\int_0^t d\sigma\|\hat{g}_i(\sigma)\|^2_{L^2(\mathbb{R}^3)}$$
$$\le C(T,\eta)\int_0^t d\sigma\|\Phi_k f(\sigma)\|^2_{L^2(\mathbb{R}^3\times\mathbb{S}^2_R)}$$
$$\times \sup_{\xi\in\mathbb{R}^3}\int_{|p|=R} dp\,(|N_{ilk}\nu_l| + |M_{ij}\omega_j\alpha_{lk}\nu_l/D^{\pm}|)^2. \tag{2.39}$$

Using Lemma 2.2, from (2.39), we obtain,

$$\begin{aligned}\|I_{3i}^{\pm}\|_{L^2(0,T;H^1(\mathbb{R}^3))} &\leq C(T,\eta)\|\Phi\|_{L^2(0,T;L^2(\mathbb{R}^3))}\|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^6))}R \\ &\quad \times \sup_{\xi\in\mathbb{R}^3}\left(\int_{|p|=R} dp\,(1+1/\sqrt{D^{\pm}})^2\right)^{1/2} \\ &\leq C(T,\eta,\|\Phi_0\|_{L^2(\mathbb{R}^3)},\|f_0\|_{L^\infty(\mathbb{R}^6)})R^{5/2}. \end{aligned} \tag{2.40}$$

In order to deal with terms $\partial_t I_{3i}^{\pm}$, we differentiate $I_{3i}^{\pm}$ with respect to time, and we obtain,

$$\begin{aligned}\partial_t I_{3i}^{\pm} &= I_{31i}^{\pm} + I_{32i}^{\pm} \\ &= +\frac{\mathrm{i}}{2}\int_0^t d\sigma \int_{\mathbb{R}^3} d\xi \int_{|p|=R} dp\, e^{\mathrm{i}\varphi_{\pm}(t,\sigma,\xi,x)}(\partial_t\eta(t,x)/|\xi| \pm \mathrm{i}\eta(t,x))\hat{g}_i(\sigma,\xi) \\ &\quad + \frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi \int_{|p|=R} dp\, e^{\mathrm{i}\xi\cdot x}\eta(t,x)\hat{g}_i(t,\xi).\end{aligned}$$

Terms $I_{31i}^{\pm}$ define Fourier integral operators with symbols in S^0, while terms $I_{32i}^{\pm}$ define pseudo-differential operators with symbols in S^{-1}. Therefore, following the same analysis that we have done for the estimate of $\|I_{3i}^{\pm}\|_{L^2(0,T;H^1(\mathbb{R}^3))}$, we obtain,

$$\|\partial_t I_{3i}^{\pm}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq C(T,\eta,\|\Phi_0\|_{L^2(\mathbb{R}^3)},\|f_0\|_{L^\infty(\mathbb{R}^6)})R^{5/2}.$$

This estimate and (2.40) lead to,

$$\|I_{3i}^{\pm}\|_{H^1([0,T]\times\mathbb{R}^3)} \leq C(T,\eta,\|\Phi_0\|_{L^2(\mathbb{R}^3)},\|f_0\|_{L^\infty(\mathbb{R}^6)})R^{5/2}. \tag{2.41}$$

2.3.4. *A priori estimate for* $I_{2i}^{\pm}$

We continue by giving an estimate for $\|I_{2i}^{\pm}\|_{L^2(0,T;H^1(\mathbb{R}^3))}$ and $\|\partial_t I_{2i}^{\pm}\|_{L^2(0,T;L^2(\mathbb{R}^3))}$. Using (2.29) and defining,

$$\hat{g}_i(\xi) = \int_{|p|\leq R} dp \frac{1}{D^{\pm}} M_{ij}(v)\omega_j \hat{f}_0(\xi,p), \tag{2.42}$$

we have,

$$\|I_{2i}^{\pm}(t)\|_{H^1(\mathbb{R}^3)} \leq \frac{1}{2}\left\|\int_{\mathbb{R}^3} d\xi \frac{e^{\mathrm{i}\varphi_{\pm}(t,0,x,\xi)}}{|\xi|}\eta(t,x)\hat{g}_i(\xi)\right\|_{H^1(\mathbb{R}^3)}.$$

Using standard results on Fourier integral operators, the Plancherel theorem, the Cauchy–Schwarz inequality for p-integration in (2.42), and Lemma 2.2, we obtain,

$$\begin{aligned}\|I_{2i}^{\pm}(t)\|_{H^1(\mathbb{R}^3)}^2 &\leq C(\eta)\|g_i\|_{L^2(\mathbb{R}^3)}^2 \leq C(\eta)\|\hat{g}_i\|_{L^2(\mathbb{R}^3)}^2 \\ &\leq C(\eta)\|f_0\|_{L^2(\mathbb{R}^6)}^2 \sup_{\xi\in\mathbb{R}^3}\int_{|p|\leq R} dp\,(M_{ij}\omega_j/D^{\pm})^2\end{aligned}$$

$$\le C(\eta)\|f_0\|^2_{L^2(\mathbb{R}^6)} \sup_{\xi\in\mathbb{R}^3}\int_{|p|\le R}\frac{dp}{D^\pm}$$

$$\le C(\eta)\|f_0\|^2_{L^2(\mathbb{R}^6)} R^4. \tag{2.43}$$

Differentiating terms $I^\pm_{2i}$ with respect to time, we obtain,

$$\partial_t I^\pm_{2i} = \frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi\, e^{\mathrm{i}\varphi_\pm(t,0,x,\xi)}(\eta(t,x)/|\xi| \pm \mathrm{i}\eta(t,x))\hat{g}_i(\xi),$$

which are Fourier integral operators with symbols in S^0. Following the same analysis that we have done for the estimate of (2.43), we finally obtain,

$$\|I^\pm_{2i}\|_{H^1([0,T]\times\mathbb{R}^3)} \le C(T,\eta,\|f_0\|_{L^2(\mathbb{R}^6)})R^2. \tag{2.44}$$

2.3.5. *A priori estimate for* $I^\pm_{1i}$

We finish by giving an estimate for $\|I^\pm_{1i}\|_{L^2(0,T;H^1(\mathbb{R}^3))}$ and $\|\partial_t I^\pm_{1i}\|_{L^2(0,T;L^2(\mathbb{R}^3))}$. Using (2.29) and defining,

$$\hat{g}_i(t,\xi) = \int_{|p|\le R} dp \frac{1}{D^\pm} M_{ij}(v)\omega_j \hat{f}(t,\xi,p), \tag{2.45}$$

we have,

$$\|I^\pm_{1i}(t)\|_{H^1(\mathbb{R}^3)} \le \frac{1}{2}\left\|\int_{\mathbb{R}^3} d\xi \frac{e^{\mathrm{i}x\cdot\xi}}{|\xi|}\eta(t,x)\hat{g}_i(t,\xi)\right\|_{H^1(\mathbb{R}^3)}.$$

Using standard results on pseudo-differential operators, the Plancherel theorem, the Cauchy–Schwarz inequality for p-integration in (2.45), and Lemma 2.2, we obtain,

$$\begin{aligned}\|I^\pm_{1i}(t)\|^2_{H^1(\mathbb{R}^3)} &\le C(\eta)\|g_i(t)\|^2_{L^2(\mathbb{R}^3)} \le C(\eta)\|\hat{g}_i(t)\|^2_{L^2(\mathbb{R}^3)}\\ &\le C(\eta)\|f(t)\|^2_{L^2(\mathbb{R}^6)}\sup_{\xi\in\mathbb{R}^3}\int_{|p|\le R} dp\,(M_{ij}\omega_j/D^\pm)^2\\ &\le C(\eta)\|f_0\|^2_{L^2(\mathbb{R}^6)}\sup_{\xi\in\mathbb{R}^3}\int_{|p|\le R}\frac{dp}{D^\pm}\\ &\le C(\eta)\|f_0\|^2_{L^2(\mathbb{R}^6)}R^4.\end{aligned}$$

We then obtain,

$$\|I^\pm_{1i}\|_{L^2(0,T;H^1(\mathbb{R}^3))} \le C(T,\eta,\|f_0\|_{L^2(\mathbb{R}^6)})R^2. \tag{2.46}$$

We now give an estimate for $\|\partial_t I^\pm_{1i}\|_{L^2(0,T;L^2(\mathbb{R}^3))}$. Differentiating terms $I^\pm_{1i}$ with respect to time, we obtain,

$$\begin{aligned}\partial_t I^\pm_{1i} &= I^\pm_{11i} + I^\pm_{12i}\\ &= -\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi\int_{|p|\le R} dp\, e^{\mathrm{i}x\cdot\xi}\partial_t\eta(t,x)\frac{1}{|\xi|}\frac{1}{D^\pm}M_{ij}(v)\omega_j\hat{f}(t,\xi,p)\\ &\quad -\frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi\int_{|p|\le R} dp\, e^{\mathrm{i}x\cdot\xi}\eta(t,x)\frac{1}{|\xi|}\frac{1}{D^\pm}M_{ij}(v)\omega_j\partial_t\hat{f}(t,\xi,p).\end{aligned}$$

Terms $I^{\pm}_{11i}$ can be estimated as (2.46). Using Vlasov equation (2.17) to re-express the term $\partial_t \hat{f}$ in $I^{\pm}_{12i}$, and using integration by parts in momentum p, terms $I^{\pm}_{12i}$ are rewritten as,

$$\begin{aligned} I^{\pm}_{12i} &= I^{\pm}_{121i} + I^{\pm}_{122i} + I^{\pm}_{123i} \\ &= \frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi \int_{|p|\le R} dp\, e^{\mathrm{i}x\cdot\xi}\eta(t,x)\frac{1}{|\xi|}\frac{1}{D^{\pm}}M_{ij}(v)\omega_j v\cdot\xi \hat{f}(t,\xi,p) \\ &\quad + \frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi \int_{|p|=R} dp\, e^{\mathrm{i}x\cdot\xi}\eta(t,x)\frac{1}{|\xi|}\frac{1}{D^{\pm}}M_{ij}(v)\omega_j \alpha_{lk}(v)\nu_l(p)\widehat{\Phi}_k(t,\xi) \underset{\xi}{*} \hat{f}(t,\xi,p) \\ &\quad - \frac{\mathrm{i}}{2}\int_{\mathbb{R}^3} d\xi \int_{|p|\le R} dp\, e^{\mathrm{i}x\cdot\xi}\eta(t,x)\frac{1}{|\xi|}\frac{\partial}{\partial p_l}\left(\frac{1}{D^{\pm}}M_{ij}(v)\omega_j\right) \\ &\quad \times \alpha_{lk}(v)\widehat{\Phi}_k(t,\xi) \underset{\xi}{*} \hat{f}(t,\xi,p). \end{aligned}$$

Terms $I^{\pm}_{121i}$ define pseudo-differential operators in space variable x, with symbols in S^0. Following the same kind of analysis that we have done for $\|I^{\pm}_{1i}\|_{L^2(0,T;H^1(\mathbb{R}^3))}$, we obtain,

$$\begin{aligned} \|I^{\pm}_{121i}\|_{L^2(0,T;L^2(\mathbb{R}^3))} &\le C(\eta)\|f(t)\|_{L^2(\mathbb{R}^6)}\left(\sup_{\xi\in\mathbb{R}^3}\int_{|p|\le R} dp\,(M_{ij}\omega_j/D^{\pm})^2\right)^{1/2} \\ &\le C(T,\eta,\|f_0\|_{L^2(\mathbb{R}^6)})R^2. \end{aligned} \tag{2.47}$$

Terms $I^{\pm}_{122i}$ define pseudo-differential operators in space variable x, with symbols in S^{-1}. Following the same kind of analysis that we have done for $\|I^{\pm}_{3i}\|_{L^2(0,T;H^1(\mathbb{R}^3))}$, we obtain,

$$\begin{aligned} \|I^{\pm}_{122i}\|_{L^2(0,T;L^2(\mathbb{R}^3))} &\le C(T,\eta)\|\Phi_k\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^6))}R \\ &\quad \times\left(\sup_{\xi\in\mathbb{R}^3}\int_{|p|=R} dp\,(M_{ij}\omega_j\alpha_{lk}\nu_l/D^{\pm})^2\right)^{1/2} \\ &\le C(T,\eta,\|\Phi_0\|_{L^2(\mathbb{R}^3)},\|f_0\|_{L^\infty(\mathbb{R}^6)})R^{5/2}. \end{aligned} \tag{2.48}$$

Terms $I^{\pm}_{123i}$ define pseudo-differential operators in space variable x, with symbols in S^{-1}. Following the same kind of analysis that we have done for $\|I^{\pm}_{1i}\|_{L^2(0,T;H^1(\mathbb{R}^3))}$, we obtain,

$$\begin{aligned} \|I^{\pm}_{123i}\|_{L^2(0,T;L^2(\mathbb{R}^3))} &\le C(T,\eta)\|\Phi_k\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^6))}R^{3/2} \\ &\quad \times\left(\sup_{\xi\in\mathbb{R}^3}\int_{|p|\le R} dp\,(\partial_{p_l}(M_{ij}\omega_j/D^{\pm})\alpha_{lk})^2\right)^{1/2} \\ &\le C(T,\eta,\|\Phi_0\|_{L^2(\mathbb{R}^3)},\|f_0\|_{L^\infty(\mathbb{R}^6)})R^{10/3}. \end{aligned} \tag{2.49}$$

Gathering estimates (2.46)–(2.49), we finally obtain,

$$\|I^{\pm}_{1i}\|_{H^1([0,T]\times\mathbb{R}^3)} \leq C(T,\eta,\|f_0\|_{L^2(\mathbb{R}^6)})R^{10/3}. \tag{2.50}$$

2.3.6. *Completion of the proof of Proposition* 2.1

Gathering estimates (2.36), (2.41), (2.44), (2.50), we obtain estimate (2.15), which completes the proof of Proposition 2.1.

2.4. *Contribution from high velocity particles*

In this section, we give an estimate for the contribution $\Phi^{>R}_{\eta}$ produced by high velocity particles. The term $\Phi^{>R}_{\eta}$ is estimated for one part in H^1 norm and for a second part in L^2 norm. Indeed, we split $\Phi^{>R}_{\eta}$ in two parts defined by,

$$\Phi^{>R}_{\eta\ell i}(t,x) = -\int_0^t d\sigma \int_{\mathbb{R}^3} d\xi\, e^{\mathrm{i}x\cdot\xi}\frac{\sin(|\xi|(t-\sigma))}{|\xi|}\widehat{J}^{>R}_{\eta\ell i}(t,\sigma,x,\xi), \quad \ell\in\{1,2\},$$

where we set,

$$\widehat{J}^{>R}_{\eta 1i}(t,\sigma,x,\xi) = \int_{|p|>R} dp\,\eta(t,x)N_{ijk}(v)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \partial_{p_j}\hat{f}(\sigma,\xi,p),$$

and,

$$\widehat{J}^{>R}_{\eta 2i}(t,\sigma,x,\xi) = \mathrm{i}\int_{|p|>R} dp\,\eta(t,x)M_{ij}(v)\xi_j\hat{f}(\sigma,\xi,p).$$

Obviously, we have $J^{>R}_{\eta i} = J^{>R}_{\eta 1i} + J^{>R}_{\eta 2i}$. Therefore, $\Phi^{>R}_{\eta\ell}$, for $\ell\in\{1,2\}$, is equivalently the solution of the vector wave equation,

$$\begin{aligned} \Box\Phi^{>R}_{\eta\ell} &= J^{>R}_{\eta\ell}, \\ \Phi^{>R}_{\eta\ell|_{t=0}} &= \Phi^{>R}_{\ell 0} = 0, \\ \partial_t\Phi^{>R}_{\eta\ell|_{t=0}} &= \Phi^{>R}_{\ell 1} = 0. \end{aligned} \tag{2.51}$$

If we assume that $J^{>R}_{\eta 1} \in L^2(0,T;L^2(\mathbb{R}^3))$, then from [30, Theorem 8.1, Sec. 8, Chap. 3] the problem (2.51) with $\ell = 1$ has a unique weak solution such that $\Phi^{>R}_{\eta 1} \in L^2(0,T;H^1(\mathbb{R}^3)) \cap L^\infty(0,T;H^1(\mathbb{R}^3)) \cap \mathscr{C}(0,T;H^1(\mathbb{R}^3))$, $\partial_t\Phi^{>R}_{\eta 1}\in L^2(0,T;L^2(\mathbb{R}^3))\cap L^\infty(0,T;L^2(\mathbb{R}^3))\cap\mathscr{C}(0,T;L^2(\mathbb{R}^3))$, and, (see estimate (8.15) in [30, Sec. 8, Chap. 3])

$$\left\|\Phi^{>R}_{\eta 1}\right\|^2_{H^1([0,T]\times\mathbb{R}^3)} \lesssim \int_0^T dt\int_0^t ds\left\|J^{>R}_{\eta 1}(s)\right\|^2_{L^2(\mathbb{R}^3)}, \tag{2.52}$$

for all finite time T. Moreover, if we assume that $J^{>R}_{\eta 2} \in L^2(0,T;H^{-1}(\mathbb{R}^3))$, then from [30, Theorems 9.3 and 9.4, Sec. 9, Chap. 3] the problem (2.51) with $\ell = 2$ has a unique weak solution such that $\Phi^{>R}_{\eta 2} \in L^2(0,T;L^2(\mathbb{R}^3))\cap L^\infty(0,T;L^2(\mathbb{R}^3))\cap$

$\mathscr{C}(0,T;L^2(\mathbb{R}^3))$, $\partial_t\Phi_{\eta 1}^{>R}\in L^2(0,T;H^{-1}(\mathbb{R}^3))\cap L^\infty(0,T;H^{-1}(\mathbb{R}^3))\cap\mathscr{C}(0,T;H^{-1}(\mathbb{R}^3))$, and, (see estimate (9.32) in [30, Sec. 9, Chap. 3])

$$\|\Phi_{\eta 2}^{>R}\|^2_{L^2([0,T]\times\mathbb{R}^3)}\lesssim\int_0^T dt\int_0^t ds\|J_{\eta 2}^{>R}(s)\|^2_{H^{-1}(\mathbb{R}^3)}, \tag{2.53}$$

for all finite time T. To estimate $\|J_{\eta 1}^{>R}\|_{L^2(0,T;L^2(\mathbb{R}^3))}$ and $\|J_{\eta 2}^{>R}\|_{L^2(0,T;H^{-1}(\mathbb{R}^3))}$, we need to know how densities of charge and current created by high velocity particles (i.e. for $|p|>R$) decrease with R in L^2 norm. This result is given by the following Lemma 2.3, which is due to the authors of [10] (see Lemma 4). Since the proof of this lemma is quite elementary and short, for the sake of completeness we reproduce it here below.

Lemma 2.3 ([10]). *Let $f(t,x,p)$ be a measurable function on $\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3$. Then, for any $\alpha\in[0,1]$, one has,*

$$\left\|\int_{|p|>R}|f|dp\right\|_{L^2([0,T]\times\mathbb{R}^3)}$$
$$\le\frac{9}{R^{\frac{3(1-\alpha)}{\alpha+3}}}\|f\|^{\frac{\alpha}{\alpha+3}}_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)}\left\|\int_{\mathbb{R}^3}\sqrt{1+|p|^2}|f|dp\right\|^{\frac{3}{\alpha+3}}_{L^{\frac{6}{\alpha+3}}([0,T]\times\mathbb{R}^3)}.$$

Proof. For all $R>0$, one has,

$$\int_{\mathbb{R}^3}|f|dp\le\int_{|p|\le R}|f|dp+\int_{|p|>R}|f|dp$$
$$\le\frac{4\pi R^3}{3}\|f\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)}+\frac{1}{R^\alpha}\int_{\mathbb{R}^3}|p|^\alpha|f|dp.$$

Taking R such that,

$$R^3\|f\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)}=\frac{1}{R^\alpha}\int_{\mathbb{R}^3}|p|^\alpha|f|dp,$$

this inequality becomes,

$$\int_{\mathbb{R}^3}|f|dp\le 9\|f\|^{\frac{\alpha}{\alpha+3}}_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)}\left(\int_{\mathbb{R}^3}|p|^\alpha|f|dp\right)^{\frac{3}{\alpha+3}}.$$

Applying this last estimate to the function $\mathbb{1}_{|p|>R}f$, we obtain,

$$\left\|\int_{|p|>R}|f|dp\right\|_{L^2([0,T]\times\mathbb{R}^3)}\le 9\|f\|^{\frac{\alpha}{\alpha+3}}_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)}\left\|\int_{|p|>R}|p|^\alpha|f|dp\right\|^{\frac{3}{\alpha+3}}_{L^{\frac{6}{\alpha+3}}([0,T]\times\mathbb{R}^3)}$$
$$\le\frac{9}{R^{\frac{3(1-\alpha)}{\alpha+3}}}\|f\|^{\frac{\alpha}{\alpha+3}}_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)}\left\|\int_{|p|>R}|p||f|dp\right\|^{\frac{3}{\alpha+3}}_{L^{\frac{6}{\alpha+3}}([0,T]\times\mathbb{R}^3)}$$
$$\le\frac{9}{R^{\frac{3(1-\alpha)}{\alpha+3}}}\|f\|^{\frac{\alpha}{\alpha+3}}_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)}\left\|\int_{\mathbb{R}^3}\sqrt{1+|p|^2}|f|dp\right\|^{\frac{3}{\alpha+3}}_{L^{\frac{6}{\alpha+3}}([0,T]\times\mathbb{R}^3)},$$

which completes the proof. ∎

2.4.1. *A priori estimate for* $J_{\eta 1}^{>R}$ *and* $\Phi_{\eta 1}^{>R}$

Here, we give an estimate of $J_{\eta 1}^{>R}$ in L^2 norm in space and time variables. Using (2.52), this estimate induces a bound for $\Phi_{\eta 1}^{>R}$ in H^1 norm. Using an integration by parts in momentum, the term $\widehat{J}_{\eta 1}^{>R}$ is rewritten as,

$$\begin{aligned}\widehat{J}_{\eta 1i}^{>R} &= \widehat{J}_{\eta 11i}^{>R} + \widehat{J}_{\eta 12i} = \widehat{J}_{\eta 11}^{>R} + \liminf_{R\to+\infty} \hat{g}_{\eta i}^{>R}\\ &= -\int_{|p|=R} dp\, \eta(t,x) N_{ijk}(v)\nu_j(v)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \hat{f}(\sigma,\xi,p)\\ &\quad + \liminf_{R\to+\infty}\int_{|p|=R} dp\, \eta(t,x) N_{ijk}(v)\nu_j(v)\widehat{\Phi}_k(\sigma,\xi) \underset{\xi}{*} \hat{f}(\sigma,\xi,p).\end{aligned}$$

We first show that $\widehat{J}_{\eta 12i} = 0$, a.e. on $\mathbb{R}_+^* \times \mathbb{R}^3$. Using Fourier transform, and the relation $N_{ijk}\nu_j\Phi_k = v_i E\cdot\nu$, we obtain,

$$\begin{aligned}g_{\eta i}^{>R}(t,x) &= \eta(t,x)\int_{|p|=R} dp\, N_{ijk}(v)\nu_j(v)\Phi_k(\sigma,x) f(\sigma,x,p)\\ &= \eta(t,x)\int_{|p|=R} dp\, E(\sigma,x)\cdot\nu(p) v_i(p) f(\sigma,x,p). \qquad (2.54)\end{aligned}$$

Using (2.54), the lower semi-continuity of the norm, the Hausdorff–Young inequality (see, e.g. [40, Sec. 7.12^a, Chap. XII]), and the Cauchy–Schwarz inequality, we obtain,

$$\begin{aligned}\|\widehat{J}_{\eta 12i}\|_{L^2(0,T;L^\infty(\mathbb{R}^3))} &\le \liminf_{R\to+\infty}\|g_{\eta i}^{>R}\|_{L^2(0,T;L^1(\mathbb{R}^3))}\\ &\le \liminf_{R\to+\infty}\left\|\eta\int_{|p|=R} dp\, E\cdot\nu v_i f\right\|_{L^2(0,T;L^1(\mathbb{R}^3))}\\ &\le C(\eta)\|E\|_{L^2(0,T;L^2(\mathbb{R}^3))}\liminf_{R\to+\infty}\left\|\int_{|p|=R} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}.\end{aligned}$$

Let us assume

$$\int_{\mathbb{R}^3} dp\,\sqrt{1+|p|^2} f \in L^{\frac{6}{\alpha+3}}([0,T]\times\mathbb{R}^3), \quad \forall\,\alpha\in[0,1]. \qquad (2.55)$$

Since $\|f\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)} \le \|f_0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} < +\infty$, using Lemma 2.3 with assumption (2.55), we obtain, for all $\delta > 0$,

$$\begin{aligned}\frac{C}{(R-\delta/2)^{\frac{3(1-\alpha)}{3+\alpha}}} &\ge \left\|\int_{|p|\ge R-\delta/2} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}\\ &= \left\|\int_{R-\delta/2}^{\infty} dr\int_{|p|=r} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}\\ &\ge \delta\left\|\frac{1}{\delta}\int_{R-\delta/2}^{R+\delta/2} dr\int_{|p|=r} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}, \qquad (2.56)\end{aligned}$$

where the constant C depends on $\|f_0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}$ and $\|\int_{\mathbb{R}^3}\sqrt{1+|p|^2}\times f\,dp\|_{L^{\frac{6}{\alpha+3}}([0,T]\times\mathbb{R}^3)}$.

Since $f_0\in L^p$ for $1\leq p\leq\infty$, using Young inequality we easily obtain the integrability property, $h(t,x,r):=\int_{|p|=r} f\,dp\in L^p_{\mathrm{loc}}(\mathbb{R}^*_+\times\mathbb{R}^3\times\mathbb{R}_+)$, for $1\leq p<\infty$. From the differentiation theorem of Lebesgue–Besicovitch (see, e.g. [39, Corollary 1, Sec. 1.1, Chap. I]) and a particular property of approximation of identity by characteristic functions (see, e.g. [39, Theorem 2 and its Corollary, Sec. 2.2, Chap. III]), this integrability property implies,

$$\frac{1}{\delta}\int_{R-\delta/2}^{R+\delta/2} dr\, h(t,x,r)\to h(t,x,R)\quad \text{a.e. on } \mathbb{R}^*_+\times\mathbb{R}^3\times\mathbb{R}_+, \tag{2.57}$$

and in $L^\infty(0,T;L^p(\mathbb{R}^3\times\mathbb{R}_+))$, for $1\leq p<\infty$ and $\forall\, T<\infty$, as δ tends to zero. From (2.56)–(2.57), for a.e. $R\in\mathbb{R}_+$, for all $\varepsilon>0$, there exists $\bar\delta_{\varepsilon,R}$ such for all $\delta\leq\delta'<\bar\delta_{\varepsilon,R}$ we have,

$$\begin{aligned}\frac{C}{(R-\delta/2)^{\frac{3(1-\alpha)}{3+\alpha}}\delta} &\geq \left\|\int_{|p|=R} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}\\ &\quad-\left|\left\|\int_{|p|=R} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}-\left\|\frac{1}{\delta'}\int_{R-\delta'/2}^{R+\delta'/2} dr\int_{|p|=r} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}\right|\\ &\geq \left\|\int_{|p|=R} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}-\varepsilon,\end{aligned}$$

that is,

$$\frac{C}{(R-\delta/2)^{\frac{3(1-\alpha)}{3+\alpha}}\delta}+\varepsilon\geq\left\|\int_{|p|=R} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}\geq 0. \tag{2.58}$$

We now take $\delta=R^{-3(1-\alpha)/(2(3+\alpha))}$ in (2.58) and let R tend to infinity while ε tends accordingly to zero. We then obtain,

$$\liminf_{R\to+\infty}\left\|\int_{|p|=R} dp\, f\right\|_{L^2(0,T;L^2(\mathbb{R}^3))}=0.$$

Therefore, we have $\|\widehat{J}_{\eta 12i}\|_{L^2(0,T;L^\infty(\mathbb{R}^3))}=0$, and $J_{\eta 12i}=0$, a.e. on $\mathbb{R}^*_+\times\mathbb{R}^3$. We now deal with the term $J^{>R}_{\eta 1}$. Using Fourier transform, the Plancherel theorem, the relation $N_{ijk}\nu_j\Phi_k=v_iE\cdot\nu$ and the Cauchy–Schwarz inequality, we obtain, for any $\varphi\in L^2(\mathbb{R}^3)$,

$$\begin{aligned}\langle J_{\eta 11i}(\sigma),\varphi\rangle_{L^2,L^2}&=\langle\widehat{J}_{\eta 11i}(\sigma),\widehat{\varphi}\rangle_{L^2,L^2}\\ &=-\int_{\mathbb{R}^3} d\xi\int_{|p|=R} dp\,\eta(t,x)N_{ijk}(v)\nu_j(v)\widehat{\Phi}_k(\sigma,\xi)\underset{\xi}{*}\hat f(\sigma,\xi,p)\widehat{\varphi}(\xi)\end{aligned}$$

$$
\begin{aligned}
&= -\eta(t,x)\int_{\mathbb{R}^3} dy \int_{|p|=R} dp\, E(\sigma,y)\cdot\nu(p)v_i(p)f(\sigma,y,p)\varphi(y)\\
&\le C(\eta)\|E(\sigma)\|_{L^2(\mathbb{R}^3)}\|\varphi\|_{L^2(\mathbb{R}^3)}\\
&\quad\times\left\|\int_{|p|=r} dp|\nu(p)v_i(p)|f\right\|_{L^\infty(0,T;L^\infty(\mathbb{R}^6))}\\
&\le C(\eta)\|E(\sigma)\|_{L^2(\mathbb{R}^3)}\|\varphi\|_{L^2(\mathbb{R}^3)}\|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^6))}R^2.
\end{aligned}
$$

Therefore, we obtain $\|J_{\eta 11i}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \le C(\eta, T, \|E_0\|_{L^2(\mathbb{R}^3)}, \|f_0\|_{L^\infty(\mathbb{R}^6)})R^2$, and from (2.52) we finally have,

$$
\|\Phi_{\eta 1}^{>R}\|_{H^1([0,T]\times\mathbb{R}^3)} \le C(\eta, T, \|E_0\|_{L^2(\mathbb{R}^3)}, \|f_0\|_{L^\infty(\mathbb{R}^6)})R^2. \tag{2.59}
$$

2.4.2. *A priori estimate for* $J_{\eta 2}^{>R}$ *and* $\Phi_{\eta 2}^{>R}$

Here, we give an estimate of $J_{\eta 2}^{>R}$ in H^{-1} norm in space variables. Using (2.53), this estimate induces a bound for $\Phi_{\eta 2}^{>R}$ in L^2 norm. Using Fourier transform, density of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ in $H^1(\mathbb{R}^3)$, the Plancherel theorem and the Cauchy–Schwarz inequality, we obtain, for any $\varphi \in H^1(\mathbb{R}^3)$,

$$
\begin{aligned}
\langle J_{\eta 2i}(\sigma), \varphi\rangle_{H^{-1},H^1} &= \langle \widehat{J}_{\eta 2i}(\sigma), \widehat{\varphi}\rangle_{\mathcal{S}',\mathcal{S}}\\
&= \mathrm{i}\int_{\mathbb{R}^3} d\xi \int_{|p|>R} dp\eta(t,x)M_{ij}(v)\xi_j\hat{f}(\sigma,\xi,p)\widehat{\varphi}(\xi)\\
&= -\eta(t,x)\int_{\mathbb{R}^3} dy\int_{|p|>R} dp\, M_{ij}(v)f(\sigma,y,p)\partial_j\varphi(y)\\
&\le C(\eta)\|\varphi\|_{H^1(\mathbb{R}^3)}\left\|\int_{|p|>R} dp\, f(\sigma)\right\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Using this inequality, estimate (2.53) and Lemma 2.3 with the assumption (2.55), we finally obtain,

$$
\begin{aligned}
&\|\Phi_{\eta 2}^{>R}\|_{L^2([0,T]\times\mathbb{R}^3)}\\
&\quad\le \frac{C(\eta,T)}{R^{\frac{3(1-\alpha)}{\alpha+3}}}\|f_0\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3)}^{\frac{\alpha}{\alpha+3}}\left\|\int_{\mathbb{R}^3}\sqrt{1+|p|^2}f\,dp\right\|_{L^{\frac{6}{\alpha+3}}([0,T]\times\mathbb{R}^3)}^{\frac{3}{\alpha+3}}\\
&\quad\le C\left(\eta, T, \|f_0\|_{L^\infty(\mathbb{R}^6)}, \left\|\int_{\mathbb{R}^3}\sqrt{1+|p|^2}f\,dp\right\|_{L^{\frac{6}{\alpha+3}}([0,T]\times\mathbb{R}^3)}\right)R^{\frac{3(\alpha-1)}{\alpha+3}}.
\end{aligned}
\tag{2.60}
$$

2.5. *Completion of the proof of Theorem* 1.1

In this section, we complete the proof of Theorem 1.1. Setting $q = 6/(\alpha+3)$, with $\alpha \in [0,1]$, assumption (2.55) is then equivalent to assumption (1.8) with

$q \in [3/2, 2]$. Now, we split the electromagnetic field Φ in two parts Φ_1 and Φ_2 where $\Phi_1 = \Phi^0 + \Phi^{<R} + \Phi_1^{>R}$ and $\Phi_2 = \Phi_2^{>R}$. Using estimates (2.14), (2.15) and (2.59), there exists a constant C_1, which depends on $\|E_0\|_{L^2(\mathbb{R}^3)}$ and $\|f_0\|_{L^2\cap L^\infty(\mathbb{R}^3)}$, such that $\|\Phi_1\|_{H^1_{\text{loc}}(\mathbb{R}_+^*\times\mathbb{R}^3)} \le C_1 R^{10/3}$. In addition, using estimate (2.60), there exists a constant C_2, which depends on $\|f_0\|_{L^\infty(\mathbb{R}^3)}$ and $\|\int\sqrt{1+|p|^2} f dp\|_{L^q(\mathbb{R}_+\times\mathbb{R}^3)}$, with $q \in]3/2, 2]$, such that $\|\Phi_2\|_{L^2_{\text{loc}}(\mathbb{R}_+^*\times\mathbb{R}^3)} \le C_2 R^{3-2q}$. Using an interpolation theorem between L^2 and H^1 (see, e.g. [5]) we obtain,

$$\|\Phi\|_{H^s_{\text{loc}}(\mathbb{R}_+^*\times\mathbb{R}^3)} \le C_1^s C_2^{1-s} R^{\frac{10}{3}s+(3-2q)(1-s)},$$

which is uniformly bounded with respect to R if $s < (2q-3)/(2q+1/3)$. Let us remark that this condition on s is slightly better than the same condition obtained in Remarks 1.2 and 1.3. We can still improve slightly this regularity by a boostraping argument. Indeed, using $\Phi \in H^s_{\text{loc}}(\mathbb{R}_+^*\times\mathbb{R}^3)$, with $s < (2q-3)/(2q+1/3)$, we can improve the estimate of terms $\|I_{4i}^\pm\|_{L^2(0,T;H^1(\mathbb{R}^3))}$. Revisiting estimates of Sec. 2.3.2, and using Hölder inequality, we obtain,

$$\begin{aligned}\|I_{4i}^\pm(t)\|^2_{H^1(\mathbb{R}^3)} &\le C(T,\eta)R^{11/3}\|\Phi f \mathbb{1}_{|p|<R}\|^2_{L^2(0,T;L^2(\mathbb{R}^3))}\\ &\le C(T,\eta)R^{11/3+3/\sigma}\|\Phi\|^2_{L^{2\sigma}([0,T]\times\mathbb{R}^3)}\|f\|^2_{L^{2\sigma'}([0,T]\times\mathbb{R}^3)},\end{aligned}$$

with $1/\sigma + 1/\sigma' = 1$. Therefore, we obtain,

$$\begin{aligned}&\|I_{4i}^\pm\|_{L^2(0,T;H^1(\mathbb{R}^3))}, \quad \|\partial_t I_{4i}^\pm\|_{L^2(0,T;H^1(\mathbb{R}^3))}\\ &\quad\le C(T,\eta,\|\Phi\|_{L^{2\sigma}([0,T]\times\mathbb{R}^3)}, \|f_0\|_{L^{2\sigma'}(\mathbb{R}^3)})R^{\frac{1}{2}(\frac{11}{3}+\frac{3}{\sigma})}.\end{aligned}$$

We now have to determine the value of σ such that $\|\Phi\|_{L^{2\sigma}([0,T]\times\mathbb{R}^3)} < +\infty$. We know that $\Phi \in H^s_{\text{loc}}(\mathbb{R}_+^*\times\mathbb{R}^3)$, with $s = (2q-3)/(2q-3+\ell_0(q))$, $\ell_0(q) = 10/3$, and $3/2 < q \le 2$. By Sobolev embeddings, we have $\Phi \in L^r_{\text{loc}}(\mathbb{R}_+^*\times\mathbb{R}^3)$, with $r = 4/[2-(2q-3)/(2q-3+\ell_0(q))]$. Thus, $\Phi \in L^{2\sigma}_{\text{loc}}(R_+^*\times\mathbb{R}^3)$, with $\sigma = r/2$, and $f \in L^\infty(\mathbb{R}_+; L^{2\sigma'}(\mathbb{R}^6))$, with $\sigma' = (2q-3+\ell_0(q))/(2q-3)$. The new dependence in R of the constant $C(R)$ is such that,

$$C(R) = \left(R^{\frac{3}{\sigma}}R^{\frac{11}{3}}\right)^{1/2} = R^{\ell_1(q)}, \quad \text{with } \ell_1(q) = \frac{31q-93/2+20\ell_0(q)}{12q-18+6\ell_0(q)}.$$

Interpolation of spaces then leads to the constraint $s\ell_1(q) + (3-2q)(1-s) < 0$ and by recurrence we obtain,

$$\Phi \in H^s_{\text{loc}}(R_+^*\times\mathbb{R}^3), \quad \text{with } s = \frac{2q-3}{2q-3+\ell_\infty(q)},$$

where $\ell_\infty(q)$ is a fixed point of the map,

$$\ell \mapsto \gamma(\ell) = \frac{31q-93/2+20\ell}{12q-18+6\ell}. \tag{2.61}$$

A straightforward study of the map (2.61), which uses standard results on quadratic polynomials, shows that there exist two fixed points $\ell_\infty^\pm(q)$ defined by,

$$\ell_\infty^\pm(q) = \frac{19}{6} + q\left(-1 \pm \sqrt{1-\frac{7}{6q}+\frac{41}{18q^2}}\right).$$

We observe that $3 < \ell_\infty^+(2) < 10/3$, $\ell_\infty^+(3/2) = 10/3$, $\ell_\infty^-(2) < 0$ and $\ell_\infty^-(3/2) = 0$. Therefore, we obtain,

$$\Phi \in H^s_{\mathrm{loc}}(R_+^* \times \mathbb{R}^3), \quad \text{with } s = \frac{2q-3}{2q-3+\ell_\infty^+(q)},$$

which completes the proof of Theorem 1.1. Of course, the best regularity result for the electromagnetic field, which has been announced in the abstract, is obtained by taking $q = 2$, namely $s = 6/(13+\sqrt{142})$.

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