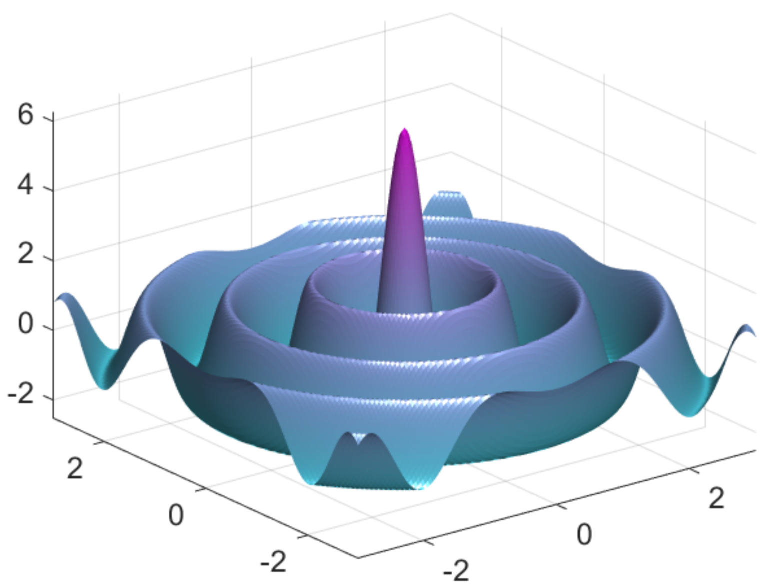
Fourier analysis course

level : L3/bachelor in physics

Eric Aristidi



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Contents

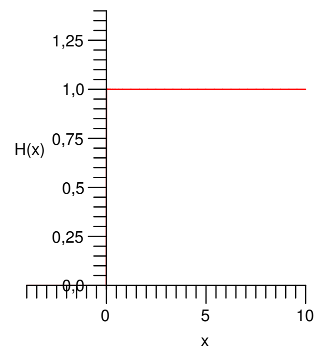
Chapter 1

Discontinuous signal— Dirac Distribution

1.1 The heaviside function $H(x)$

$H(x)$ is a function defined as follows for a real number x :

$$H(x) = 0 \text{ if } x < 0$$
$$H(x) = 1 \text{ if } x > 0 \qquad (1.1)$$



We sometimes speak of a "step function" or a "unit step". $H(x)$ in not defined at $x = 0$[1]. However, we will sometimes make a continuous extension when necessary ; for example the function $H(x) + H(-x)$ is 1 $\forall x \neq 0$ but is not defined at $x = 0$. We will therefore extend it (by giving it the value 1) at $x = 0$ so that it is continuous on $\mathbb{R}$. We sometimes define $H(x)$ by making use of any function f, denoted as *test function* by the relation

$$\int_{-\infty}^{\infty} f(x)\, H(x)\, dx \; = \; \int_{0}^{\infty} f(x) dx \tag{1.2}$$

it is the definition in the sense of distributions (though distribution theory will not be presented in this course). In physics, $H(t)$ is sometimes used for functions of the time t which are zero-valued for $t < 0$. For example, a stone released at $t = 0$ from the altitude z_0 without initial speed has a motion described by the altitude function $z(t) = z_0 - \frac{1}{2} g t^2 H(t)$. Another example in optics if the transmission coefficient of a half-plane.

1. There exist other determinations at $x = 0$ according to various authors. We can thus find $H(0) = 0$, $H(0) = 1$ ou $H(0) = \frac{1}{2}$.

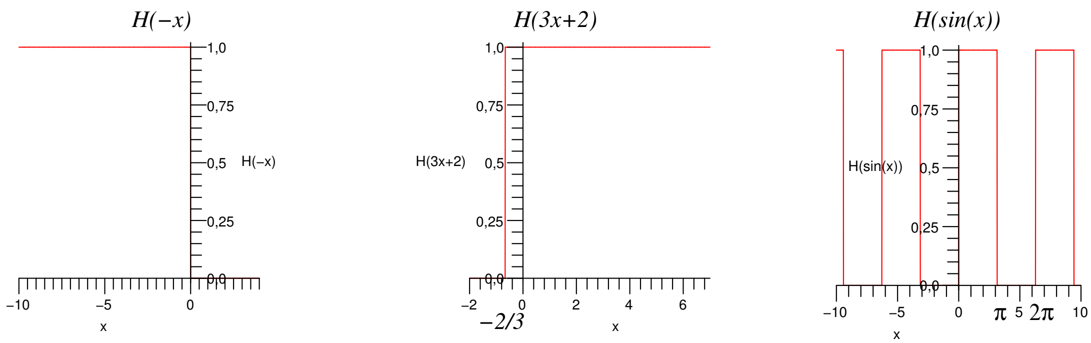
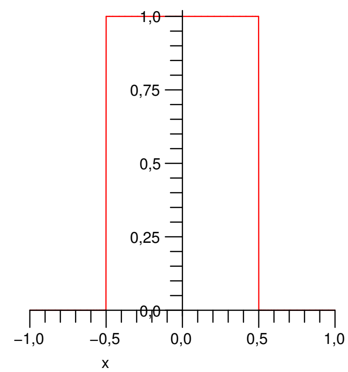


FIGURE 1.1 – Example of functions using the Heaviside unit step.

1.2 The rectangular function $\Pi(x)$

The rectangular function (or *rectangle function*, or *gate function*) $\Pi(x)$ is also a piecewise discontinuous function : :

$$\Pi(x) = 1 \text{ if } |x| < \tfrac{1}{2}$$
$$\Pi(x) = 0 \text{ otherwise} \tag{1.3}$$

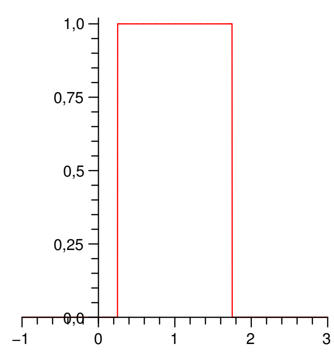


It is not defined at its two edges $x = \pm\frac{1}{2}$ (but continuous extensions may be applied if necessary). This function has a width 1, i.e. it is nonzero on an interval of width 1. $\Pi(x)$ is linked to the Heaviside function by

$$\Pi(x) = H(x + \frac{1}{2}) - H(x - \frac{1}{2}) \tag{1.4}$$

In physics, the rectangular function is sometimes used to define signals of finite duration. Hence the function

$$f(t) = \Pi\left(\frac{t-b}{a}\right) \tag{1.5}$$



corresponds to a gate of width a (or duration a if t is a time) centered at $t = b$. Another example is the charge density of a sphere of diameter D centered at the origin, of uniform charge density ρ_0. Its charge density in the whole space can be expressed, in spherical coordinates (r designates the distance to the origin), by the function $\rho(r) = \rho_0\,\Pi(\frac{r}{D})$.

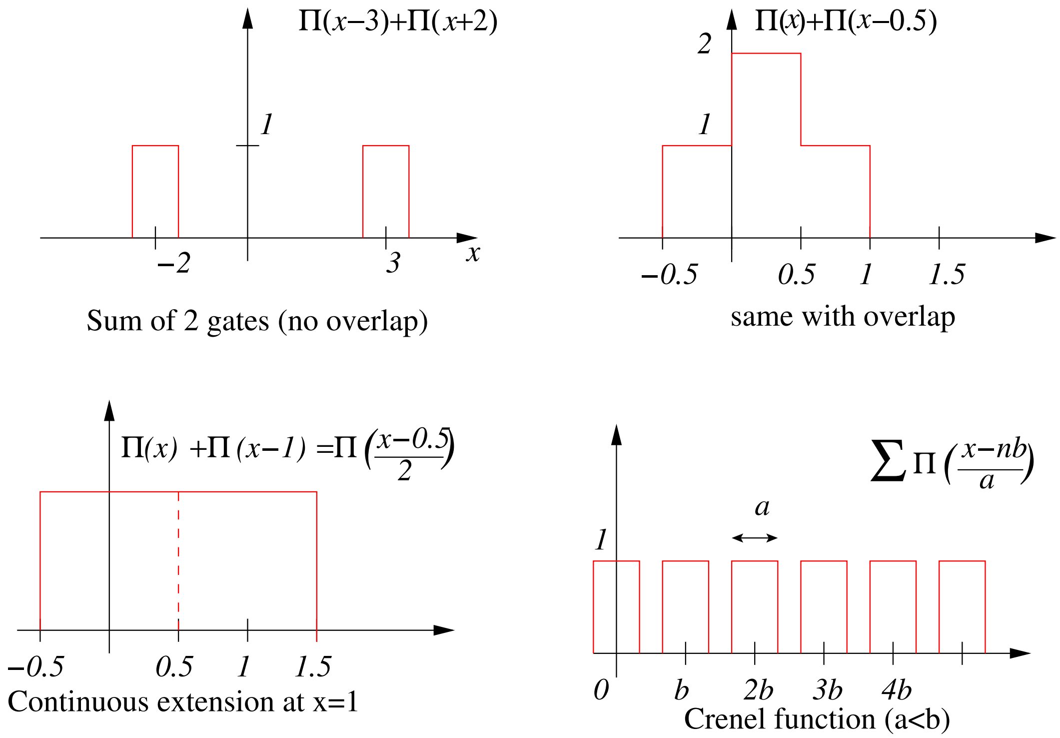


FIGURE 1.2 – Examples gate functions and sum of gates

2D rectangular function

We consider functions of 2 variables x and y. The quantity

$$f(x,y) = \Pi(x) \;=\; \Pi(x)\,\mathbf{1}(y)$$

describes a strip of width 1 parallel to the y axis : it is invariant by translation along y. The notation $\mathbf{1}(y)$ stands for a function which value is 1 whatever y.
A two dimensional rectangle function of width a in the x direction and b in the y direction expresses as

$$f(x,y) = \Pi\left(\frac{x}{a}\right)\,\Pi\left(\frac{y}{b}\right) \tag{1.6}$$

It is represented in perspective plot in 1.3. This function is often used in optics to express the transmission coefficient of rectangular slits.

2D circular function

We consider the following quantity :

$$f(x,y) = \Pi\left(\frac{\rho}{d}\right) \tag{1.7}$$

with $\rho = \sqrt{x^2+y^2}$. Its value is one for $\rho < \frac{d}{2}$, i.e. inside a disc of diameter d (see Fig. 1.4). This function is used in optics to describe transmission coefficient of circular diaphragms.

1.3 Dirac distribution $\delta(x)$

1.3.1 Heuristic approach

We consider the function

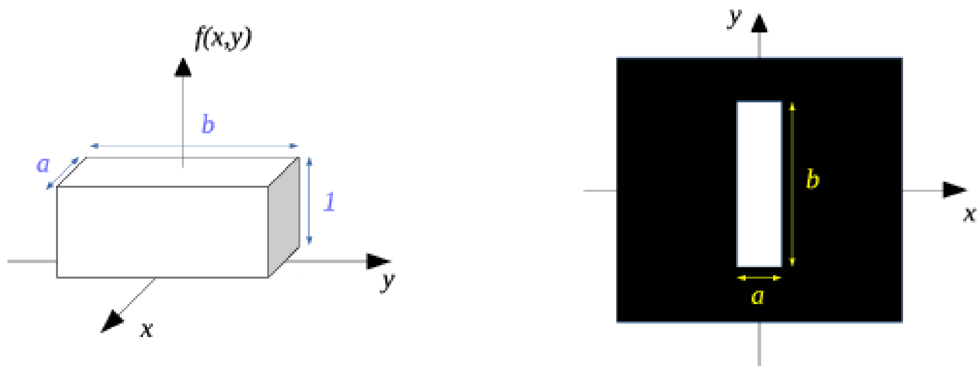


FIGURE 1.3 – 2D rectangular function $f(x, y) = \Pi\left(\frac{x}{a}\right)\Pi\left(\frac{y}{b}\right)$ of width a in the x direction and b in the y. Left : perspective plot as a function of x and y. Right : gray-level representation in the (x, y) plane.

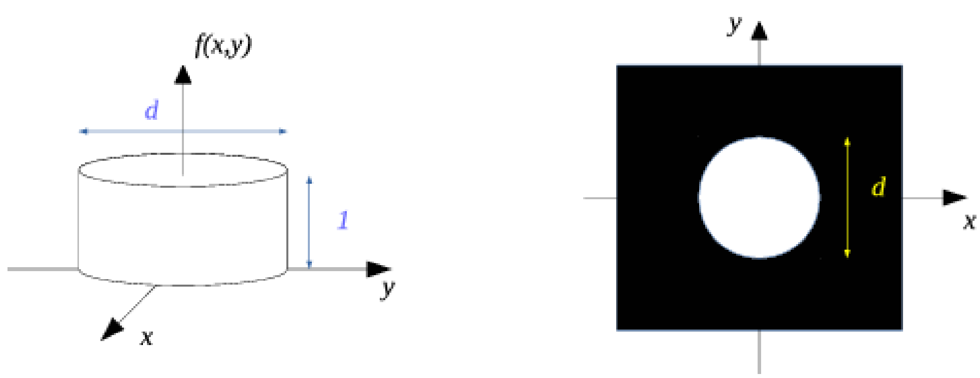
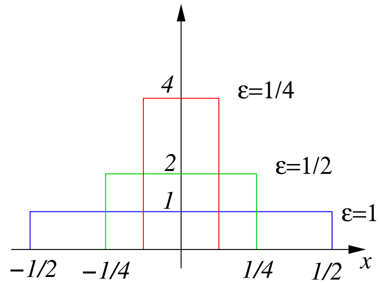


FIGURE 1.4 – 2D circular function $f(x, y) = \Pi\left(\frac{\rho}{d}\right)$ of diameter d. Left : perspective plot as a function of x and y. Right : gray-level representation in the (x, y) plane.

$$g_\epsilon(x) = \frac{1}{\epsilon} \prod\left(\frac{x}{\epsilon}\right) \tag{1.8}$$



It is a rectangle of width ϵ and height $1/\epsilon$. Its integral is 1 :

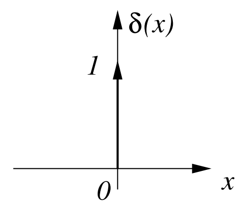
$$\int_{-\infty}^{\infty} g_\epsilon(x)\, dx \;=\; 1 \tag{1.9}$$

When $\epsilon \to 0$ this function has a width which tends to 0 and a height to infinity, but its integral is always 1. We will call *Dirac distribution* and we will denote $\delta(x)$ this limit :

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \prod\left(\frac{x}{\epsilon}\right) \tag{1.10}$$

δ has a zero width, an infinite height and an integral 1. It is often denoted as "Dirac impulse". The graph of δ will be represented by an upward arrow, of height 1, centered at $x = 0$.

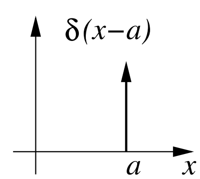
The height of the arrow is 1 to mean that the integral of δ is 1. For the graph of 2δ, it will be an arrow of height 2. **Be careful, do not confuse δ with the Kronecker function which takes values 0 and 1, and is the discrete analog of the Dirac delta distribution**.



It is easy to see that $N\,\delta(x)$ (with $N \neq 0$) has an integral N, but there is a problem if $N = 0$. We shall admit [2] that $0\,.\,\delta(x) = 0$.

We also have the following properties :

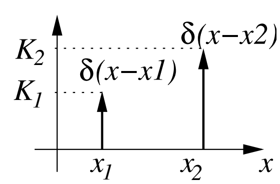
— Translation : $\delta(x-a)$ is 0 everywhere excepted at $x = a$ (Dirac impulse located at $x = a$)



— The sum

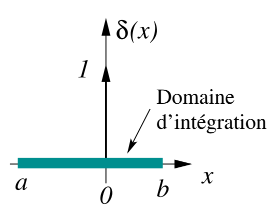
$$K_1\,\delta(x - x_1) + K_2\,\delta(x - x_2) \tag{1.11}$$

is a double peak, of integrals K_1 and K_2 located at x_1 and x_2. If $x_1 \neq x_2$ there is no superposition of the two peaks (the first one is zero everywhere except at x_1, the second is zero everywhere except at x_2).



— The integral of $\delta(x)$ is 1 on any interval $[a, b]$ so that $a < 0$ and $b > 0$:

$$\int_a^b \delta(x)\, dx \;=\; 1 \tag{1.12}$$



In physics, the δ distribution is used to describe impulses, such as a very intense and very brief force (a kick in a ball). Point charges are also described by δ distributions : an infinite density in a zero volume but a finite total charge.

2. In physics, δ does not exist, we will rather be dealing with very localized functions of very high but finite amplitude such as g_ϵ : the product $0\,.\,g_\epsilon(x) = 0$ is no more a problem in this case, and tends towards 0 when $\epsilon \to 0$

1.3.2 Fundamental property — Definition of δ

Let f a function of a real variable x, not infinite at $x = 0$, and integrable over $\mathbb{R}$. We denote as I the following quantity :

$$I = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx \tag{1.13}$$

To approach this integral, we make use of the function g_ϵ whose δ is the limit when $\epsilon \to 0$:

$$I = \int_{-\infty}^{\infty} f(x) \lim_{\epsilon\to 0} g_\epsilon(x)\,dx \tag{1.14}$$

and we permute limit and integral (ensuring that the integral converges in the interval) :

$$I = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f(x)\,g_\epsilon(x)\,dx \tag{1.15}$$

as the function $g_\epsilon(x)$ is zero outside the interval $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$, the integration domain is reduced to this interval. We obtain :

$$I = \lim_{\epsilon\to 0} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} f(x)\,\frac{1}{\epsilon}\,dx \tag{1.16}$$

the variable change $x = \epsilon y$ gives :

$$I = \lim_{\epsilon\to 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\epsilon y)\,dy \tag{1.17}$$

and its limit is

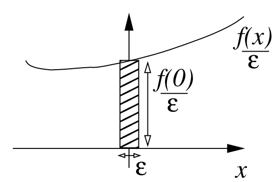
$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(0)dy \;=\; f(0) \tag{1.18}$$

We eventually obtain :

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx \;=\; f(0)} \tag{1.19}$$

which is in fact the true definition of the Dirac distribution δ. Warning, this is only valid if $f(0)$ exists, writing $\frac{\delta(x)}{x}$ does not make sense. δ is not an ordinary function, in the sense that it is not defined by its value at each point (saying that it is infinite at 0 is not enough). δ is in fact defined by the integral under a curve : we speak of *distribution* and not of function.

On a graph, when $\epsilon \to 0$, the integral $\int f(x)\,g_\epsilon(x)\,dx$ corresponds to the surface of the rectangle of width ϵ and height $f(0)/\epsilon$. This surface is $f(0)$.



We can also remark, since $f(x)\,\delta(x)$ is zero everywhere except at $x = 0$ where it is infinite, that this product can be assimilated to a Dirac distribution. And since its integral is $f(0)$, we have

$$f(x)\,\delta(x) \;=\; f(0)\,\delta(x) \tag{1.20}$$

which generalizes in :

$$\boxed{f(x)\,\delta(x-a) \;=\; f(a)\,\delta(x-a)} \tag{1.21}$$

1.3.3 Some properties of δ

δ may be built with any function

Let g a function non singular at the origin, of integral 1 on $\mathbb{R}$ (not necessary a gate function). It can be shown that

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\, g\left(\frac{x}{\epsilon}\right) \tag{1.22}$$

To obtain this, let's consider a test function f, integrable on $\mathbb{R}$, and calculate the integral :

$$I = \int_{-\infty}^{\infty} f(x) \lim_{\epsilon \to 0} \frac{1}{\epsilon} g\left(\frac{x}{\epsilon}\right) dx \tag{1.23}$$

by reasoning similar to that of the previous paragraph, we show that $I = f(0)$. Therefore, from the definition of δ given in the section1.3.2, the quantity $\lim_{\epsilon \to 0} \frac{1}{\epsilon} g\left(\frac{x}{\epsilon}\right)$ identifies to $\delta(x)$.

δ is even

This is trivial : $\delta(-x) = \delta(x)$

Dimension of $\delta(x)$

If x is a physical quantity having a dimension (for exemple a length). The integral

$$\int_{-\infty}^{\infty} \delta(x)\, dx \tag{1.24}$$

is the dimensionless number 1. As a result, the product $\delta(x)\, dx$ is dimensionless and

$$[\delta(x)] \;=\; [x]^{-1} \tag{1.25}$$

$\longrightarrow$ *The delta impulse $\delta(x)$ is homogeneous to the inverse of its argument x.*

Contraction or dilatation

Let a a nonzero real number. What is the value of $\delta(ax)$? It is easy to see that $\delta(ax)$ is 0 everywhere except at $x = 0$ where it in infinite. It can therefore be assimilated to a Dirac distribution and can be written $K\,\delta(x)$. The constant K is simply the integral of $\delta(ax)$ on $\mathbb{R}$. Let's calculate K :

$$K = \int_{-\infty}^{\infty} \delta(ax)\, dx \tag{1.26}$$

the variable change $y = ax$ gives

$$K = \int_{-\infty}^{\infty} \delta(y)\, \frac{1}{|a|}\, dy \tag{1.27}$$

the absolute value $|a|$ comes from the change of sign of the limits if $a < 0$. We obtain $K = \frac{1}{|a|}$. And we have this important property :

$$\boxed{\delta(ax) \;=\; \frac{1}{|a|}\, \delta(x)} \tag{1.28}$$

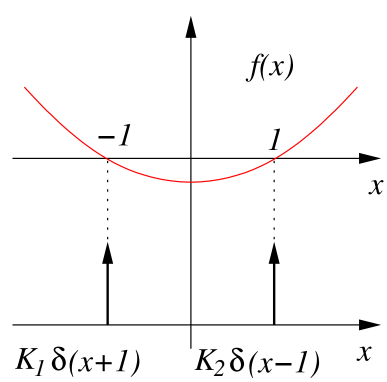
Distribution $\delta(f(x))$

Example :

Let $f(x) = x^2 - 1$. This function has two zeros at $x \pm 1$. The quantity $\delta(f(x))$ is zero where its argument $f(x) \neq 0$ and infinite where $f(x) = 0$, i.e. at $x \pm 1$. It can be written as the sum of 2 Diracs peaks centered at -1 and 1 :

$$\delta(f(x)) = K_1 \delta(x+1) + K_2 \delta(x-1) \tag{1.29}$$

the constants K_1 et K_2 are calculated below.



To calculate K_2 we integrate $\delta(x^2-1)$ around the point $x=1$. We choose an integration domain $0<\epsilon \ll 1$ around $x=1$. We obtain :

$$K_2 = \int_{1-\epsilon}^{1+\epsilon} \delta(x^2-1)\,dx \tag{1.30}$$

than we make a Taylor expansion of f around $x=1$:

$$f(x) \simeq f(1) + (x-1)\,f'(1) \;=\; (x-1)\,f'(1) \tag{1.31}$$

it comes

$$K_2 \simeq \int_{1-\epsilon}^{1+\epsilon} \delta((x-1)\,f'(1))\,dx \tag{1.32}$$

so, if $f'(1) \neq 0$

$$K_2 \simeq \frac{1}{|f'(1)|}\int_{1-\epsilon}^{1+\epsilon} \delta((x-1))\,dx \;=\; \frac{1}{|f'(1)|} \tag{1.33}$$

the same reasoning applied to the calculation of K_1 (near the point $x=-1$) gives

$$K_1 = \frac{1}{|f'(-1)|} \tag{1.34}$$

so that :

$$\delta(x^2-1) \;=\; \frac{1}{|f'(1)|}\,\delta(x-1) \;+\; \frac{1}{|f'(-1)|}\,\delta(x+1) \tag{1.35}$$

Generalisation We consider a fonction f having N roots at $x=x_i$ and nonzero derivatives $f'(x_i)$ at $x=x_i$. The quantity $\delta(f(x))$ is zero everywhere except when $f(x)=0$, i.e. for $x=x_i$, where it is infinite. As for the example above, it behaves as a sum of Delta peaks located at $x=x_i$. The calculation of integrals under each peak can me made following the same process as above. We obtain :

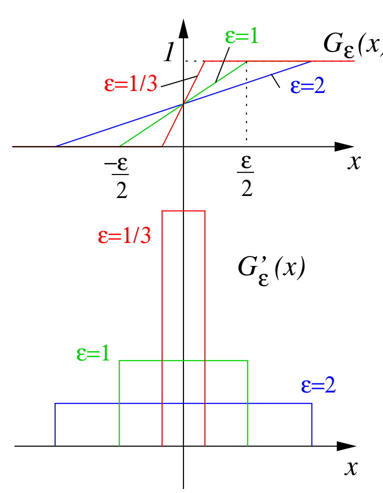
$$\boxed{\delta(f(x)) \;=\; \sum_{i=1}^{N} \frac{1}{|f'(x_i)|}\,\delta(x-x_i)} \tag{1.36}$$

1.3.4 Derivative of discontinuous signals

Derivative of the Heaviside function

Heuristic approach The Heaviside function $H(x)$ has a slope 0 everywhere except at $x=0$ where it is infinite. Can we deduce that its derivative is a δ distribution ?
To check this, we can define $H(x)$ by the limit $\epsilon \to 0$ of the function $G_\epsilon(x)$ define as :

$$G_\epsilon(x) = \begin{cases} 0 \text{ if } x \leq -\dfrac{\epsilon}{2} \\ \dfrac{x}{\epsilon} + \dfrac{1}{2} \text{ if } |x| \leq \dfrac{\epsilon}{2} \\ 0 \text{ if } x \geq \dfrac{\epsilon}{2} \end{cases} \tag{1.37}$$



This function has a slope $\frac{1}{\epsilon}$ which tends to infinity, inside an interval of width $\epsilon \to 0$ around the origine. It is easy to see that its derivative is a gate function :

$$G'_\epsilon(x) = \frac{1}{\epsilon} \prod\left(\frac{x}{\epsilon}\right) \tag{1.38}$$

which tends towards $\delta(x)$ when $\epsilon \to 0$. And since $G_\epsilon(x) \to H(x)$ when $\epsilon \to 0$, we can deduce that $H'(x) = \delta(x)$.

Proof Let's calculate the primitive of δ : $\int_{-\infty}^{x} \delta(t)\, dt$. This integral is 0 if $x < 0$, 1 if $x > 0$ and not defined at $x = 0$: this is exactly the definition of $H(x)$. Hence :

$$H(x) = \int_{-\infty}^{x} \delta(t)\, dt \tag{1.39}$$

and

$$\frac{dH}{dx} = \delta(x) \tag{1.40}$$

this generalises the notion of derivative to discontinuous functions.

Derivative of a function discontinuous at the origin

We consider a piecewise function f. We call $f_+(x)$ its value for $x > 0$ and $f_-(x)$ its value for $x < 0$. It is assumed that the two functions do not connect at the origin, the discontinuity jump is $h = f_+(0) - f_-(0)$. We want to have an expression of the derivative of f at the discontinuity. As for the Heaviside, the function slope is infinite at $x = 0$ and we expect to see a δ distribution appear.
Let's write the general expression of f, using the following compact form :

$$f(x) = H(x) f_+(x) + H(-x) f_-(x) \tag{1.41}$$

then apply the usual rules of derivation to find the derivative of f :

$$f'(x) = H(x) f'_+(x) + \delta(x) f_+(x) + H(-x) f'_-(x) - \delta(-x) f_-(x) \tag{1.42}$$

which gives after using the property $f(x)\delta(x) = f(0)\delta(x)$

$$f'(x) = [H(x) f'_+(x) + H(-x) f'_-(x)] + \delta(x)\,[f_+(0) - f_-(0)] \tag{1.43}$$

the first bracket above is the usual derivative of f at any point $x \neq 0$. The second term concerns the behavior at the origin. We find the result that

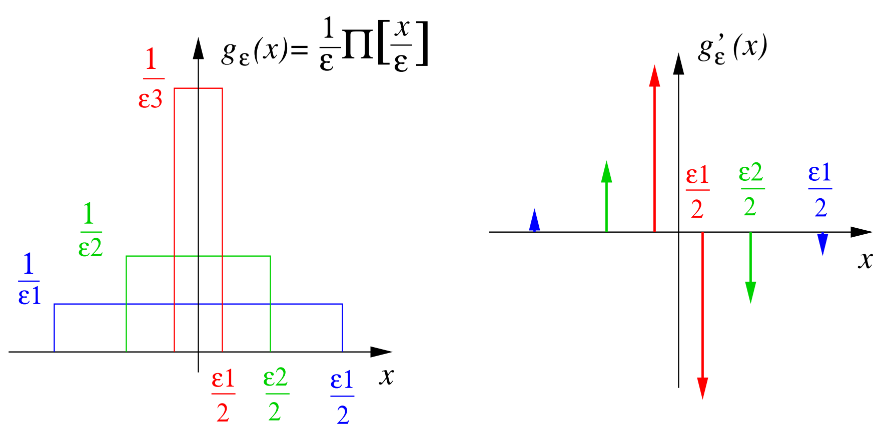
$\longrightarrow$ *the derivative at the origin of a function f presenting a discontinuity h is equal to $h\delta(x)$*

In physics, a sudden change in speed (elastic shock for example) can be treated using this formalism. The velocity discontinuity corresponds to a δ distribution in the acceleration [3].

Derivatives of δ

Heuristic approach We have introduced the Dirac distribution as the limit of an infinitely high and narrow gate function whose integral is 1 :

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \prod \left(\frac{x}{\epsilon} \right) \tag{1.44}$$



3. It is of course a modeling ; in reality the speed never changes instantaneously and the acceleration is actually a very high and very narrow peak

In the same idea, we can imagine approaching the derivative of δ by the derivative of this gate (drawing above). The derivative of $\Pi(x/\epsilon)$ is $\delta(x+\epsilon/2)-\delta(x-\epsilon/2)$. When $\epsilon \to 0$ it tends towards the superposition of 2 Diracs centered at 0, of opposite sign, each of infinite integral. Of course it is simply a representation, there is no question of calculating the limit (the result would lead to writing $\frac{1}{x}\delta(x)-\frac{1}{x}\delta(x)$ which doesn't make sense).
The proper definition of δ' is indeed, in the sense of distributions :

$$\int_{-\infty}^{\infty} f(x)\,\delta'(x)\,dx = -f'(0) \tag{1.45}$$

where f is any test function derivable at $x=0$. Similarly we define the derivative of order m by

$$\int_{-\infty}^{\infty} f(x)\,\delta^{(m)}(x)\,dx = (-1)^m f^{(m)}(0) \tag{1.46}$$

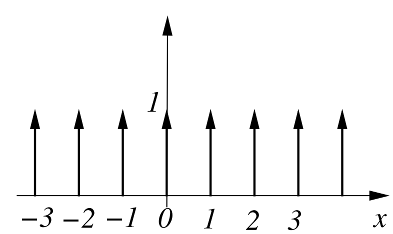
In electromagnetism we use the distributions δ' to model the dipole moments.

1.4 The Dirac comb $Ш(x)$

The Dirac comb is composed of a periodic succession of δ impulses :

$$Ш(x) = \sum_{n=-\infty}^{\infty} \delta(x-n) \tag{1.47}$$

its period is 1. The comb is of capital importance in signal processing, it is the tool which makes it possible to formally describe the sampling operation. It is also the basis of the representation of all periodic phenomena as we will see in the next chapter (convolution)



Scale change The comb $Ш(x)$ has a period 1. How is the comb of period $a>0$ written, i.e. whose "teeth" are δ peaks of integral 1 spaced by a? This quantity, that we will denote as $Ш_a(x)$, can be expressed as :

$$Ш_a(x) = \sum_{n=-\infty}^{\infty} \delta(x-na) \tag{1.48}$$

Using the relation $\delta(ax)=\frac{1}{|a|}\delta(x)$ we can write

$$Ш_a(x) = \frac{1}{|a|}\sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{a}-n\right) \tag{1.49}$$

and finally

$$Ш_a(x) = \frac{1}{|a|}\, Ш\left(\frac{x}{a}\right) \tag{1.50}$$

The classic error when writing a comb of period a is to forget the term $\frac{1}{|a|}$ in factor : it is this which ensures that the Diracs have integral 1 (if you forget it, their integral is a). It also contains the dimension of the comb ($Ш_a(x)$ is indeed homogeneous to $1/x$ and to $1/a$ whereas $Ш(x/a)$ is dimensionless).

1.5 Two-dimensional Dirac distribution

We define the two-dimensional Dirac distribution by

$$\delta(x,y) \;=\; \delta(x).\delta(y) \tag{1.51}$$

it is sometimes denoted by ${}^2\delta(x,y)$. It is zero everywhere except at the point $(x=0,y=0)$ where it is infinite and verifies the property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)\,dx\,dy \;=\; 1 \tag{1.52}$$

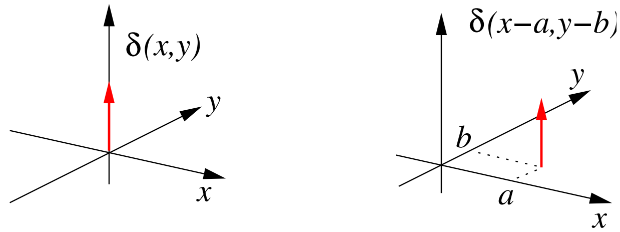
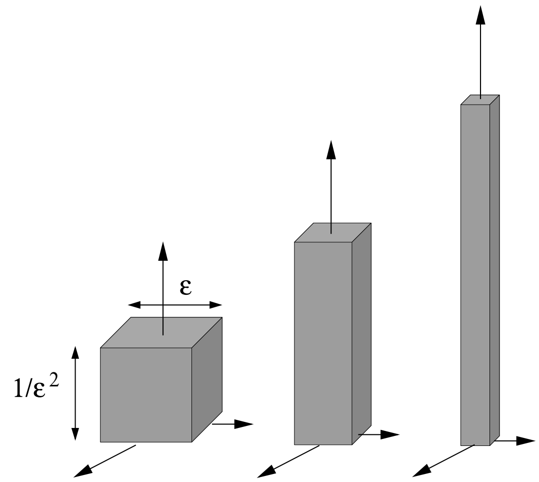


FIGURE 1.5 – Perspective representation of a two-dimensional Dirac peak. On the left $\delta(x, y)$ is centered at the origin. On the right $\delta(x-a, y-b)$ is centered at the point of coordinates (a, b)

As with its one-dimensional counterpart, $\delta(x, y)$ can be thought of as the limit of a 2D gate function, i.e. a rectangular parallelepiped with area ϵ^2 and height $1/\epsilon^2$:

$$\delta(x,y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \prod\left(\frac{x}{\epsilon}\right) \prod\left(\frac{y}{\epsilon}\right) \tag{1.53}$$



2D rectangle function of side ϵ and integral 1. When $\epsilon \to 0$ it becomes infinitely high and infinitely thin, its integral is always 1. It tends towards $\delta(x, y)$.

The two-dimensional Dirac is for example used in optics to model the transmission coefficient of a very small diameter diaphragm ("pin-hole" in English). We can also define an N-dimensional function $\delta(\vec{r})$. The classic example is the point charge q located at $\vec{r}_0$ whose charge density is

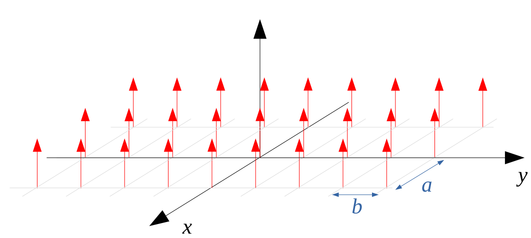
$$\rho(\vec{r}) = q\,\delta(\vec{r} - \vec{r}_0) \tag{1.54}$$

which is zero everywhere except at $\vec{r} = \vec{r}_0$, and whose integral is q.

2D Dirac comb

The 2D Dirac comb (sometimes denoted as "Dirac brush") is the product of 2 combs in directions x and y :

$$\underset{a}{Ш}(x).\,\underset{b}{Ш}(y) = \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \delta(x - na, y - pb) \tag{1.55}$$



with a and b the periods in x and y directions.

Chapter 2

The convolution

2.1 Definition

2.1.1 Definition

Let f and g two functions integrable on $\mathbb{R}$. We call *convolution product* of f by g the following integral :

$$h(x) \;=\; (f * g)(x) \;=\; \int_{-\infty}^{\infty} f(x')g(x-x')dx' \tag{2.1}$$

or, more simply, $h = f * g$. It is a functional operation, it acts on the two functions f and g and returns a function h. The notation $h(x) = f(x) * g(x)$ is incorrect because $f(x)$ denotes a number and not the function f, but it is sometimes used out of habit or simplicity .

2.1.2 Physical significance

This is the superposition integral of two functions of x' : $f(x')$ and $g(x - x') = g(-(x' - x))$. The latter is simply the function g, flipped horizontally and shifted to the point x. The operation can be schematized by the example of the figure 2.1.

Particular case : convolution by a gate function Let $g(x) = \frac{1}{a}\prod\left(\frac{x}{a}\right)$ a gate function of width a and integral 1. The convolution of any function f by g writes as :

$$h(x) \;=\; \frac{1}{a}\int_{-\infty}^{\infty} f(x')\prod\left(\frac{x-x'}{a}\right)dx' \;=\; \frac{1}{a}\int_{x-a/2}^{x+a/2} f(x')dx' \tag{2.2}$$

it is a *moving average*, ie an average value of f over an interval of width a around the point x. This operation has the effect of attenuating the rapid fluctuations of f as shown in figure 2.2 and is often used in signal processing to reduce the noise.

2.2 Properties of the convolution

2.2.1 The convolution is a commutative product

Convolution is a commutative product between two functions, in the sense that it has the following properties

Internal law : the convolution of two functions is a function

Associativity : $(f * g) * h = f * (g * h)$. Proof :

$$\begin{aligned}
((f * g) * h)(x) &= \int_{x'=-\infty}^{\infty} (f * g)(x')h(x-x')dx' \\
&= \int_{x'=-\infty}^{\infty} h(x-x')\int_{y=-\infty}^{\infty} f(y)g(x'-y)dydx' = \int_{y=-\infty}^{\infty} f(y)\int_{x'=-\infty}^{\infty} g(x'-y)h(x-x')dx'dy \\
&= \int_{y=-\infty}^{\infty} f(y)\int_{z=-\infty}^{\infty} g(z)h(x-y-z)dzdy == \int_{y=-\infty}^{\infty} f(y)(g*h)(x-y)dy \\
&= (f * (g * h))(x)
\end{aligned}$$

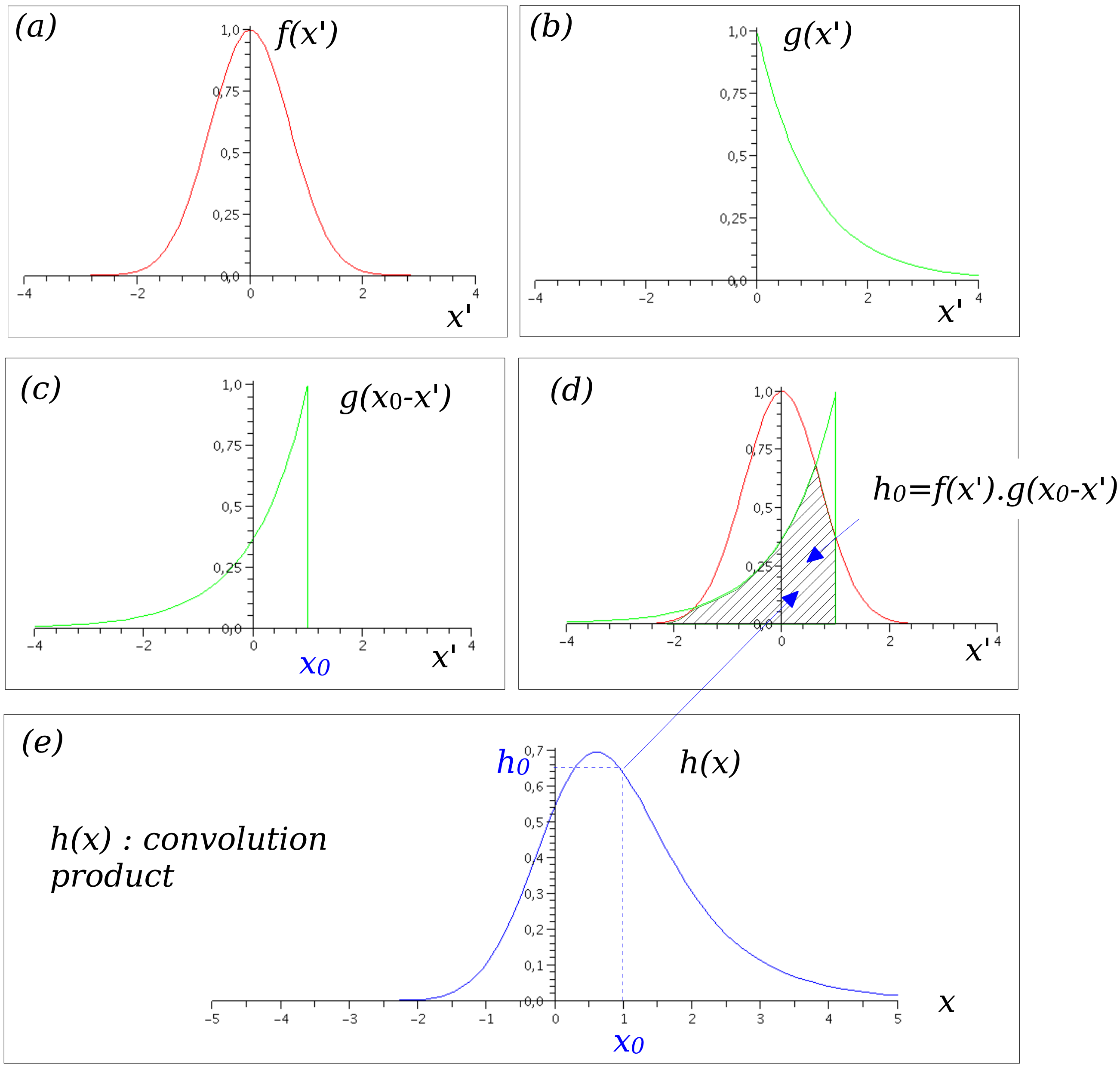


FIGURE 2.1 – Convolution of two functions f and g. (a) : graph of $f(x')$. (b) : graph of $g(x')$. (c) : graph of $g(x-x')$ for a particular value x_0 of x (the presence of the - sign in front of x' has the effect of reversing the x axis). (d) : product of the two functions $f(x').g(x_0 - x')$: the hatched overlapping area $h_0 = h(x_0) = \int_{-\infty}^{\infty} f(x')g(x_0 - x')dx'$ is the value of the convolution product $(f * g)(x)$ for $x = x_0$. (e) : when we repeat the operation for all possible values of x, we obtain the whole function h.

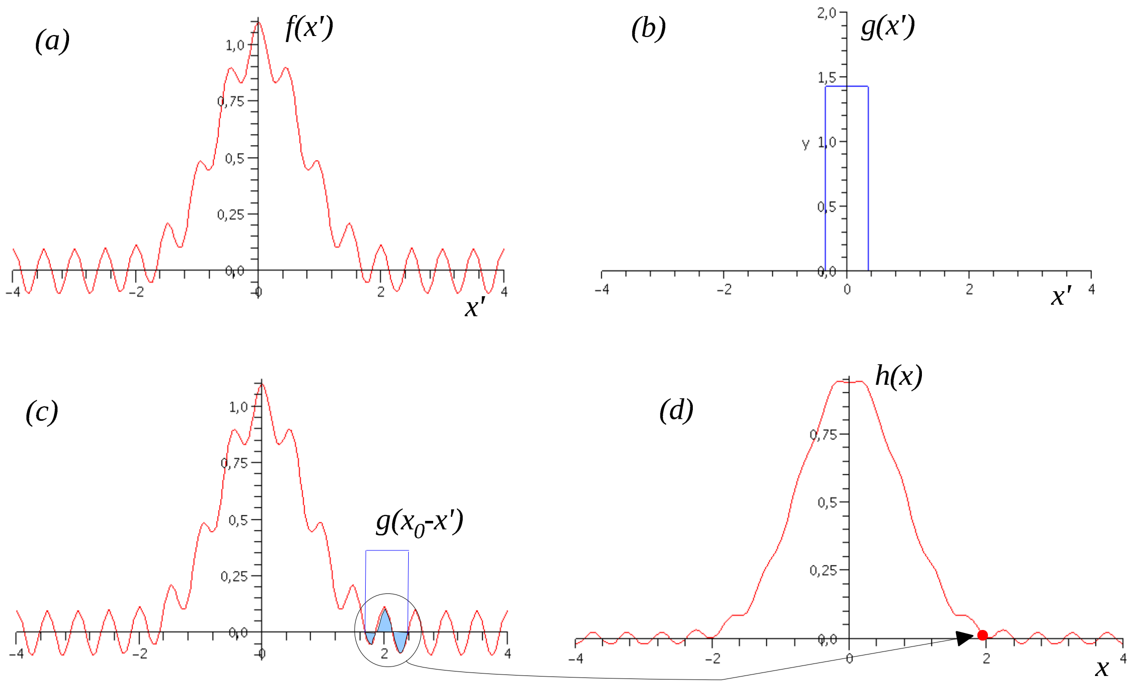


FIGURE 2.2 – Effect of a moving average (convolution by a gate function of width a) on a function f presenting rapid oscillations. (a) : the function f, (b) : the gate (of width $a = 0.8$ in this example), (c) : the function f and the area of the product $f(x').g(x_0 - x')$ with $x_0 = 2$ on the example. (d) : result of the convolution. The convolution has the effect of attenuating these oscillations by averaging the values of f over the interval of width a (the parts of the curve above and below the average over the interval compensate each other).

Commutativity : $f * g = g * f$. The proof is trivial by variable change

Neutral element : this is the δ distribution. In effect,

$$\begin{aligned}(f * \delta)(x) &= \int_{x'=-\infty}^{\infty} f(x')\delta(x - x')dx' \\ &= \int_{x'=-\infty}^{\infty} f(x)\delta(x - x')dx' \;=\; f(x)\int_{x'=-\infty}^{\infty} \delta(x - x')dx' \\ &= f(x)\end{aligned}$$

we can therefore write that $f * \delta = f$. The physical meaning of this property is to represent a function by a "sum of impulses". The function $f(x) = (f * \delta)(x) = \int_{x'=-\infty}^{\infty} f(x')\delta(x - x')dx'$ can be seen as a continuous sum of Dirac pulses $F(x') = f(x')\delta(x - x')$ located at $x' = x$ and weighted by $f(x')$.

2.2.2 Other properties

Linearity

On a :

- $f * (g + h) = f * g + f * h$
- Let a any constant :, $f * (ag) = a(f * g)$

By generalisation, one obtains : $f * \sum a_n g_n = \sum a_n (f * g_n)$ with a_n a set of constants and g_n a set of functions.

Convolution by 1

We denote here $1(x)$ the function of value 1 for all x. The result of the convolution of any function f by $\mathbf{1}$ gives :

$$(\mathbf{1} * f)(x) = \int_{x'=-\infty}^{\infty} f(x')\mathbf{1}(x - x')dx' = \int_{x'=-\infty}^{\infty} f(x')dx' \tag{2.3}$$

So the integral of f on $\mathbb{R}$ can be written as a convolution of f by the function $\mathbf{1}$. This may sound "far-fetched" but it may turn out to be practical, as in the proof of the following property.

Integral of a convolution

The integral on $\mathbb{R}$ of $f * g$ is equal to the product of the integrals of f et g :

$$\int f * g = \mathbf{1} * (f * g) = (\mathbf{1} * f) * g = \int f \,.\mathbf{1} * g = \int f \,. \int g$$

using the notation $\int f$ for the integral of f over $\mathbb{R}$ (which is a constant equals to $\int f\,.1$).

Translation

To translate (change origin) a convolution product $f * g$, one just shifts one of the two functions f or g. Thus (using improper, but convenient notation) :

$$(f * g)(x + a) = f(x + a) * g = f * g(x + a) \tag{2.4}$$

with a a real constant. the demonstration is trivial by simple variable change.

Derivation

To derive a convolution product $f * g$, just differentiate one of the two functions f or g (if you differentiate both, you get the second derivative of the convolution) :

$$(f * g)' = f' * g = f * g' \tag{2.5}$$

This property can be proved using the previous one by writing the derivative f' as the limit of the rate of change of f.

Change of scale

For any real constant λ we have the property :

$$(f * g)(\lambda x) = |\lambda| f(\lambda x) * g(\lambda x) \tag{2.6}$$

which can also be proved using a variable change.

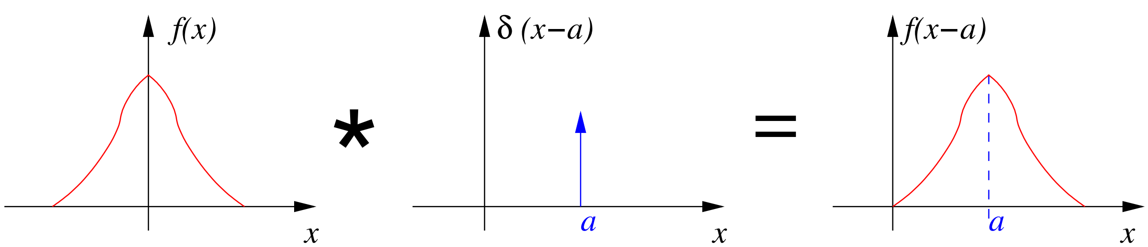
Convolution by $\delta(x - a)$

This is an important property in terms of physical significance. Thus, using the ‘translation property we can write

$$f * \delta(x - a) = (f * \delta)(x - a) = f(x - a) \tag{2.7}$$

and we obtain an interesting result : to translate a function of a quantity a, one convolves it with a Dirac peak $\delta(x - a)$. We will often write this incorrect (but practical) formula :

$$\boxed{f(x) * \delta(x - a) = f(x - a)} \tag{2.8}$$



Convolution by a Dirac comb

Using the previous property it immediately comes

$$(f * \underset{a}{\text{Ш}})(x) = f(x) * \sum_{n=-\infty}^{\infty} \delta(x - na) = \sum_{n=-\infty}^{\infty} f(x - na) \tag{2.9}$$

To convolve f by a comb is to “periodize” f, i.e. is to create a periodic function from an infinity of replicas of f centered on the “teeth” of the comb, and sum all these replicates. The operation is illustrated by figure 2.3 and

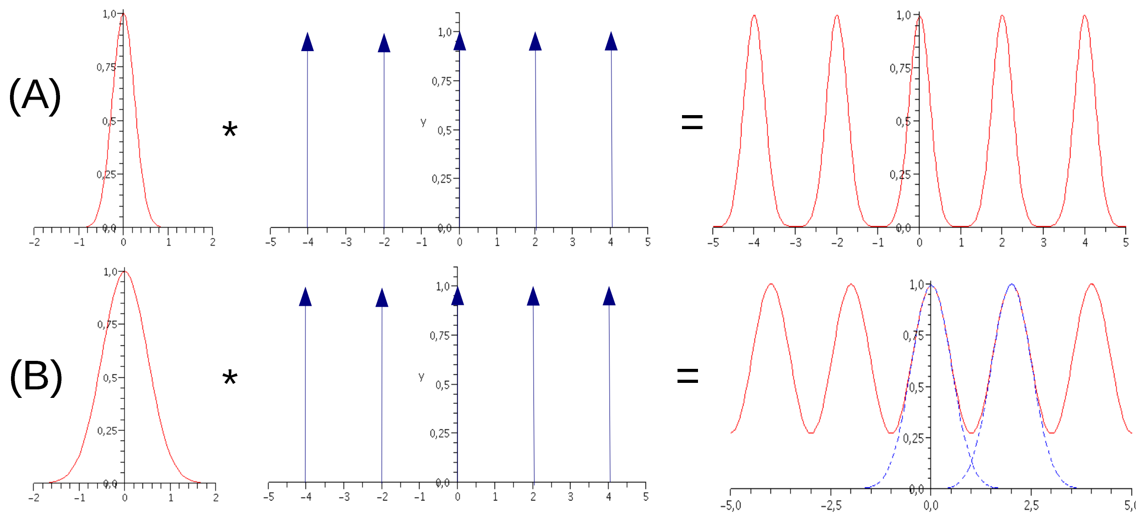


FIGURE 2.3 – The convolution of a function by a comb has the effect of periodizing the function. (A) : on the left the (Gaussian) function, in the center the comb (of period 2 in this example) and on the right the result of the convolution which is a periodic function of the same period as the comb, and whose pattern is the Gaussian. (B) : same thing with a larger Gaussian : the result of the convolution shows an overlap between the different patterns.

shows in particular that if f has support limited to the interval $[-\frac{a}{2}, \frac{a}{2}]$ then the replicas of f are disjoint and the pattern of the periodic function $f * \text{Ш}_a$ is f. Otherwise the different replicas of f overlap and sum up.
Any periodic function f of period a can thus be written in the form of a convolution of a pattern ϕ by a comb of period a. The pattern is the value of f limited to the interval $[-\frac{a}{2}, \frac{a}{2}]$, i.e.

$$\phi(x) = f(x) \prod \left(\frac{x}{a}\right) \tag{2.10}$$

We can for example write

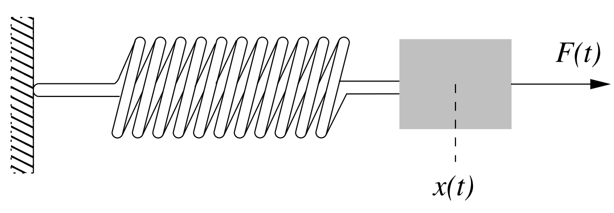
$$\cos(x) = \left[\cos(x) \prod \left(\frac{x}{2\pi}\right)\right] * \underset{2\pi}{\text{Ш}}(x) \tag{2.11}$$

2.3 Application to the solution of linear differential equations

2.3.1 Example

We consider a mass attached to a spring. We exert on the mass a force $F(t)$, and we are interested in the elongation $x(t)$ of the spring with the following initial conditions :

$$\begin{aligned} x(0) &= x_0 \\ x'(0) &= 0 \end{aligned}$$



The differential equation of the motion of the mass is that of a harmonic oscillator, it is written

$$x''(t) + \omega^2 x(t) = F(t) \tag{2.12}$$

with ω the oscillator's natural frequency. It is a linear equation (if we multiply $F(t)$ by a constant a then the solution $x(t)$ is also multiplied by a). We are going to show that the solution of such an equation can be written as a convolution between two functions : $F(t)$ (second member) and a function $R(t)$ called "impulse response" (or "point-spread function").
We are first interested in the case where the force is of the $\delta(t)$ type, ie a very intense force for a very short time (an impulse). We call $R(t)$ the corresponding solution (elongation), it obeys the equation of the oscillator. It comes

$$R''(t) + \omega^2 R(t) = \delta(t) \tag{2.13}$$

then we convolve the two members of the equation above by the function $F(t)$:

$$F * R'' + \omega^2 F * R = F * \delta \tag{2.14}$$

and we use (i) the equality $F * \delta = F$ and (ii) the property of derivation of a convolution product : $x * y' = (x * y)'$. We obtain

$$(F * R)'' \quad + \quad \omega^2 \quad (F * R) \quad = F \tag{2.15}$$

This equation is identical to the equation satisfied by x

$$x'' \quad + \quad \omega^2 \quad x \quad = F \tag{2.16}$$

Since the solution must be unique, we necessarily have

$$x = F * R \tag{2.17}$$

The impulse response $R(t)$ describes the motion of the mass when an impulse force (which is zero for $t > 0$) is applied to it : $R(t)$ is therefore, when $t > 0$, the *solution of the equation without second member*. The exact calculation of $R(t)$ will be detailed in paragraph 2.3.3

2.3.2 Generalisation

Consider a physical system governed by a linear differential equation

$$a_0 y + a_1 y' + a_2 y'' + \ldots + a_n y^{(n)} = F \tag{2.18}$$

and R its impulse response, solution of

$$a_0 R + a_1 R' + a_2 R'' + \ldots + a_n R^{(n)} = \delta \tag{2.19}$$

the reasoning of the previous paragraph applies and we also have the convolution relation

$$y = F * R \tag{2.20}$$

it is a quite remarkable property and which makes it possible to replace the sometimes laborious resolution of a differential equation by an integral calculation. This only works if the equation is linear, as shown by the example below :

Case of a nonlinear equation : let's consider the example

$$x''(t) + kx^2(t) = F(t) \tag{2.21}$$

with k a constant. We call R the solution of

$$R''(t) + kR^2(t) = \delta(t) \tag{2.22}$$

convolve the two members of this equation by F, it comes

$$(F * R)'' + kF * R^2 = F \tag{2.23}$$

which compares to the equation

$$x''(t) + kx^2(t) = F(t) \tag{2.24}$$

but we can no longer identify x here with $F * R$ because of the square term : $x^2 \neq F * R^2$. In this case the solution of the equation **is not put** *in the form of a convolution.*

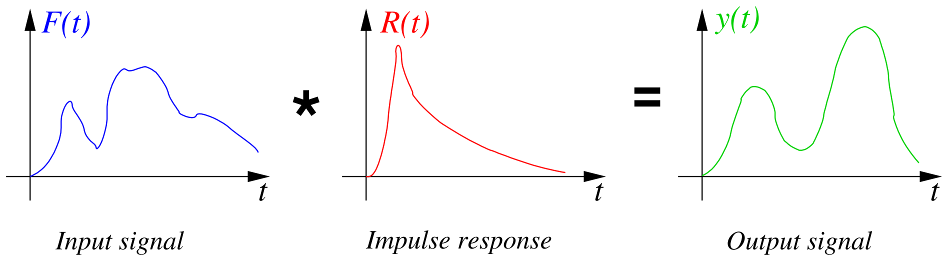
Vocabulary Writing the solution of the equation in the form of a convolution makes it possible to separate two contributions :

- An external contribution $F(t)$ (the exciting force in the example of the spring)
- A contribution specific to the physical system : the impulse response $R(t)$ (which depends on the stiffness and the mass in the example of the spring)

The mere knowledge of the impulse response $R(t)$ makes it possible to calculate the solution for any second member, so that one does not need to know what the physical system is made of if one knows its impulse response : it could be treated as a "black box". The following vocabulary, inspired by the field of signal processing, is sometimes used :

- the physical system (spring+mass for example) is called *linear system* or *filter*

— the second member of the equation is the *input signal*
— the solution $y(t)$ to the differential equation is called *response* or *output signal*
— the relation $y = F * R$ is called *input–output relation*.
and the behavior of the system can be schematized by the drawing below



We speak of *causal system* when the impulse response $R(t) = 0$ for $t < 0$. In this case the variable t designates time. The physical meaning is quite simple to understand : imagine $R(t < 0) \neq 0$ in the spring example. An impulse type force applied at time $t = 0$ would then cause the mass to move ($R(t)$ is the elongation of the spring) at $t < 0$ i.e. before force is applied. Such a system would violate the principle of causality.

2.3.3 Calculation of the impulse response : example of the spring

Let's take the equation of the harmonic oscillator from the paragraph 2.3.1 :

$$x''(t) + \omega^2 x(t) = F(t) \tag{2.25}$$

The impulse response $R(t)$ obeys the differential equation

$$R''(t) + \omega^2 R(t) = \delta(t) \tag{2.26}$$

There are several methods to calculate R. We can perform a Fourier transformation of the previous equation, which has the effect of transforming the differential equation into a simple linear equation, as we will see in the next chapter.
We propose here a more traditional method (general solution of the equation without second member + particular solution). As the $\delta(t)$ of the second member is difficult to handle, we introduce the primitive G of the impulse response :

$$G = \int R\,dt \tag{2.27}$$

with the condition $G(t \leq 0) = 0$ to respect causality. Then we integrate the equation 2.26 with respect to time :

$$R'(t) + \omega^2 \int R(t)\,dt = \int \delta(t)\,dt \tag{2.28}$$

which also writes as

$$G''(t) + \omega^2 G(t) = H(t) \tag{2.29}$$

with $H(t)$ the Heaviside distribution. For $t > 0$ the equation is thus written

$$G''(t) + \omega^2 G(t) = 1 \tag{2.30}$$

which is easy to solve. The solution of the equation without second member is

$$G_0(t) = A\,\cos(\omega t) + B\,\sin(\omega t) \tag{2.31}$$

with A and B integration constants. A particular constant solution is $G_1 = \frac{1}{\omega^2}$. The condition $G(0) = 0$ gives $A = -\frac{1}{\omega^2}$, so that

$$G(t) = \frac{1}{\omega^2}(1 - \cos(\omega t)) + B\,\sin(\omega t) \tag{2.32}$$

The impulse response is the derivative of G. The constant B vanishes because of the condition $R(0) = 0$ (spring at rest at $t \leq 0$). We finally have, for $t > 0$

$$R(t) = \frac{1}{\omega}\sin(\omega t) \tag{2.33}$$

and $R(t) = 0$ for $t < 0$. So :

$$R(t) = \frac{H(t)}{\omega}\sin(\omega t) \tag{2.34}$$

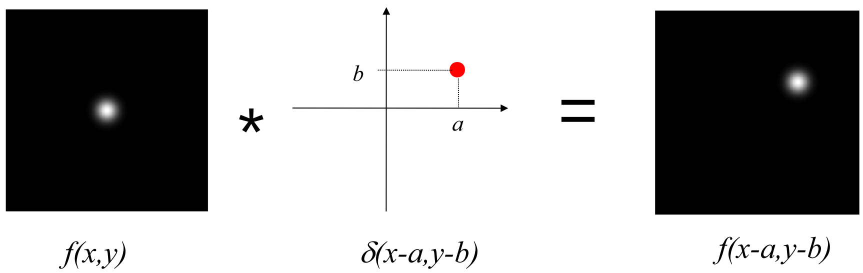


FIGURE 2.4 – Property of translation of the convolution (Eq. 2.36) : a function $f(x,y)$ (on the left) centered at the origin is convolved by a Dirac impulse centered at $(x=a, y=b)$. The result (on the right) is the shifted function $f(x-a, y-b)$ centered at $(x=a, y=b)$.

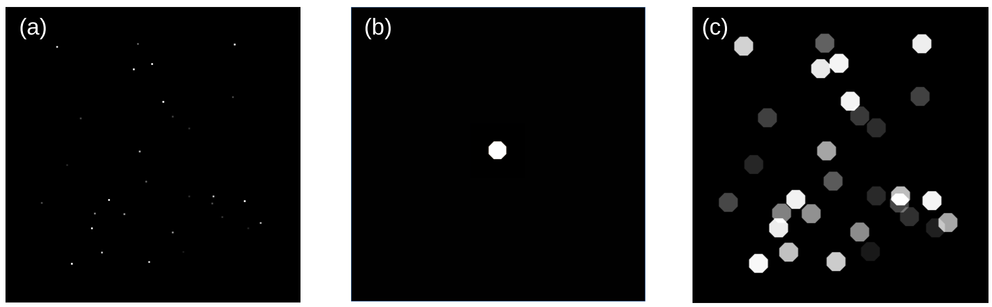


FIGURE 2.5 – Illustration of Eq. 2.37. (a) gray-scale plot of a sum of 2D Dirac impulses with different amplitudes. (b) gray-scale plot of the point-spread function $f(x,y)$ ($f(x,y)=1$ inside an octogonal domain, 0 elsewhere). (c) result of the 2D convolution of the two functions. As predicted by Eq. 2.37, the result is a sum of shifted PSFs (each impulse of the sum is replaced by the PSF, with the same amplitude A_n) When PSFs overlap, the result is the sum of overlapping terms.

2.4 2D convolution

The 2D convolution between two functions of (x,y) is

$$h(x,y) \;=\; (f*g)(x,y) \;=\; \iint_{-\infty}^{\infty} f(x',y')g(x-x',y-y')dx'\,dy' \tag{2.35}$$

Note that we use the same symbol $*$ for 1D and 2D convolutions, but the two operations are different (single integral for 1D, double integral for 2D). The 2D impulse response is sometimes denoted as "point-spread function". The 2D convolution has a lot of applications in image processing ; for example convolving an image $f(x,y)$ by a 2D rectangle function will blur the image.
Most of the properties apply to 2D convolution. In particular this one :

$$f(x,y) * \delta(x-a, y-b) = f(x-a, y-b) \tag{2.36}$$

which is illustrated by the Figure 2.4. A translation of a function inside the (x,y) plane can be expressed as a convolution by a shifted 2D Dirac impulse. This is the origin of the name "point-spread function" (PSF) for the impulse response at 2D (a 2D Dirac impulse is a infinitely sharp point in the (x,y) plane, and the convolution transforms this point into a larger function f).
A corollar of this property is :

$$f(x,y) * \sum_n A_n \delta(x-x_n, y-y_n) = \sum_n A_n f(x-x_n, y-y_n) \tag{2.37}$$

which is illustrated by Figs. 2.5 and 2.6.

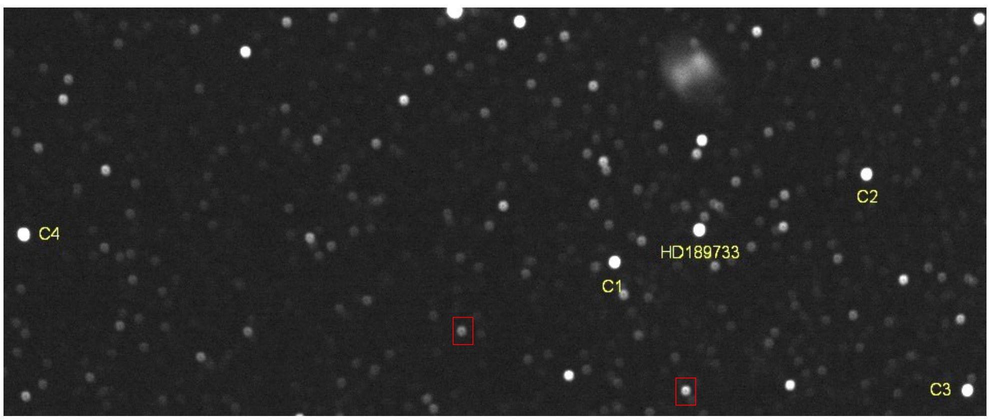


FIGURE 2.6 – Image of portion of sky with a defocused optics : each star has the shape of a small disc, with a central obstruction. This is a typical illustration of a 2D convolution as in Fig. 2.6. The perfect image $f(x, y)$ is composed of a sum of 2D impulses (ideal image of a point-source). It is convolved by a Point-Spread function $g(x, y)$ which is the small disc (two examples are in the red boxes). The fuzzy objet on the top right is the Dumbbell nebula, which is also convolved by the PSF (so that every point of the nebula is replaced by the PSF, resulting in a blurred image).

Chapter 3

Fourier transform

3.1 Definition

3.1.1 Definition

Let f be a function of real variable with real or complex values :

$$\begin{array}{rcccc} f & : & \mathbb{R} & \to & \mathbb{C} \\ & & x & \mapsto & f(x) \end{array} \tag{3.1}$$

We call *Fourier transform* (or FT) of f the integral

$$\boxed{\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-2i\pi\nu t}\, dt} \tag{3.2}$$

with ν real. This integral exists if f is integrable on $\mathbb{R}$. We speak of direct space to describe $f(t)$, and of Fourier space to describe $\hat{f}(\nu)$. We will sometimes use the following notations :

$$\hat{f} = \mathcal{F}[f] = \mathrm{FT}[f] \tag{3.3}$$

ou

$$\hat{f}(\nu) = \mathcal{F}_\nu[f] = \mathrm{FT}_\nu[f] \tag{3.4}$$

As for the convolution it is a functional operation and the writing $\hat{f}(\nu) = \mathcal{F}_\nu[f(t)]$ is incorrect...but we use it anyway for convenience and/or habit.
The quantity ν is called *conjugate variable* of t. Its dimension is inverse of that of t

$$[\nu] = [t]^{-1} \tag{3.5}$$

so if t is a time (in seconds), ν is a frequency (in seconds^{-1}). And if t is a position (in meters), then ν is a spatial frequency (in meters^{-1}). Thus the FT is a mathematical operation making it possible to transform a function depending on time into a function depending on the frequency.
It should be noted that other definitions of the Fourier transform exist. For example :

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

or

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

The passage from one definition to the other is done simply by a change of variable. Here we will work exclusively with the definition of the equation 3.2.

3.1.2 Fourier transform of real-valued functions

We are interested in the case of real-valued functions f of a real variable t. We will distinguish the case of even and odd functions :

Case of a real and even function f These functions verify $f(t) = f(-t)$. In this case its f is written as follows :

$$\begin{aligned}
\hat{f}(\nu) &= \int_{-\infty}^{\infty} f(t)\, e^{-2i\pi\nu t}\, dt \\
&= \int_{-\infty}^{0} f(t)\, e^{-2i\pi\nu t}\, dt \;+\; \int_{0}^{\infty} f(t)\, e^{-2i\pi\nu t}\, dt \\
& \downarrow \text{ variable change } y = -t \\
&= \int_{+\infty}^{0} f(-y)\, e^{2i\pi\nu y}\, d(-y) \;+\; \int_{0}^{\infty} f(t)\, e^{-2i\pi\nu t}\, dt \\
&= \int_{+\infty}^{0} f(y)\, e^{2i\pi\nu y}\, d(-y) \;+\; \int_{0}^{\infty} f(t)\, e^{-2i\pi\nu t}\, dt \\
&= \int_{0}^{\infty} f(y)\, e^{2i\pi\nu y}\, d(y) \;+\; \int_{0}^{\infty} f(t)\, e^{-2i\pi\nu t}\, dt \\
&= 2\int_{0}^{\infty} f(t)\, \cos(2\pi\nu t)\, dt
\end{aligned} \tag{3.6}$$

This last integral is called *cosine transform*. In particular, it has the following property :

$$\boxed{f \text{ real and even} \Longleftrightarrow \hat{f} \text{ real and even}}$$

Case of a real and odd function f which satisfies $f(t) = -f(-t)$. The same reasoning allows us to write

$$\hat{f}(\nu) = -2i \int_{0}^{\infty} f(t)\, \sin(2\pi\nu t)\, dt \tag{3.7}$$

The integral $2\int_0^\infty f(t)\, \sin(2\pi\nu t)\, dt$ is called *sine transform*. It is also shown that

$$\boxed{f \text{ real and odd} \Longleftrightarrow \hat{f} \text{ imaginary and odd}}$$

Case of any real function f : it always breaks down into an even (f_p) and an odd part (f_i) :

$$f(t) = f_p(t) + f_i(t) \tag{3.8}$$

with $2f_p(t) = f(t) + f(-t)$ and $2f_i(t) = f(t) - f(-t)$. The Fourier transform of real functions satisfies the following property (we say that they are *hermitian*)

$$\boxed{f \text{ real} \Longleftrightarrow \hat{f} \text{ has an even real part and an odd imaginary part}}$$

which can be written in the following compact way

$$\hat{f}(-\nu) = \overline{\hat{f}(\nu)} \tag{3.9}$$

with the notation $\bar{z}$ = complex conjugate of z.

3.1.3 Examples

Rectangle function

Let $f(t) = \Pi(t)$. Its FT writes as :

$$\begin{aligned}
\hat{f}(\nu) &= \int_{-\infty}^{\infty} \Pi(t)\, e^{-2i\pi\nu t}\, dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2i\pi\nu t}\, dt \;=\; -\frac{1}{2i\pi\nu}\left[e^{-2i\pi\nu t}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{\sin(\pi\nu)}{\pi\nu}
\end{aligned} \tag{3.10}$$

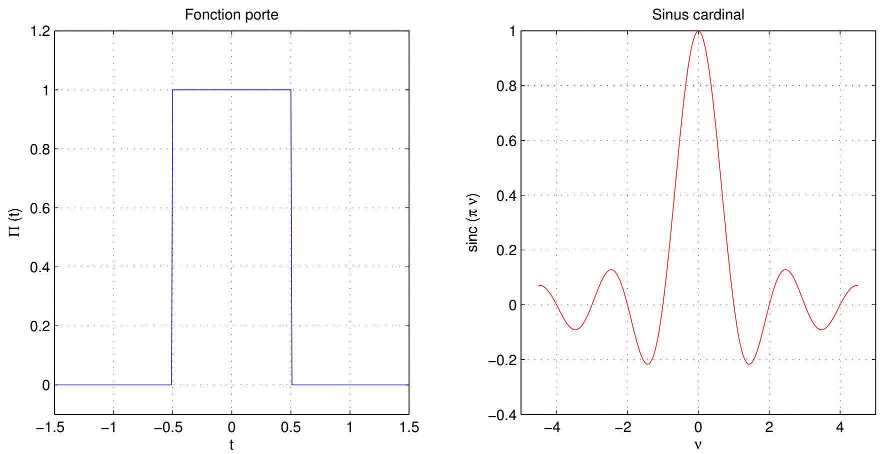


FIGURE 3.1 – Rectangle function (left) and its Fourier transform (right)

This function $\frac{\sin(\pi\nu)}{\pi\nu}$ is knows as *cardinal sine* and denoted $\mathrm{sinc}(\pi\nu)$ [1]. It has the property of vanishing for integer ν, hence the name of *cardinal*. We will remember that

$$\boxed{f(t) = \Pi(t) \iff \hat{f}(\nu) = \mathrm{sinc}(\pi\nu)} \tag{3.11}$$

The graph of the two functions f and $\hat{f}$ is shown in figure 3.1. Note that $\hat{f}(\nu)$ is here a real and even function since $f(t)$ is real and even.

Gaussian function

Let $f(t) = e^{-\pi t^2}$. Its FT writes as :

$$\begin{aligned} \hat{f}(\nu) &= \int_{-\infty}^{\infty} e^{-\pi t^2}\, e^{-2i\pi\nu t}\, dt \\ &\downarrow \quad \text{introduce the beginning of the square } (t+i\nu)^2 \\ &= e^{-\pi\nu^2} \int_{-\infty}^{\infty} e^{-\pi(t+i\nu)^2}\, dt \end{aligned} \tag{3.12}$$

The calculation of the integral $\int_{-\infty}^{\infty} e^{-\pi(t+i\nu)^2}$ is done by the method of residues, we show that it is equal to 1. So it comes that $\hat{f}(\nu) = e^{-\pi\nu^2}$. We find the known result that the FT of a Gaussian is a Gaussian. We will remember that

$$\boxed{f(t) = e^{-\pi t^2} \iff \hat{f}(\nu) = e^{-\pi\nu^2}} \tag{3.13}$$

Laplace function

Consider the function $f(t) = e^{-|t|}$. This is a function known as *Laplace's law* [2] in the domain of probability. Its FT is written :

$$\begin{aligned} \hat{f}(\nu) &= \int_{-\infty}^{\infty} e^{-|t|}\, e^{-2i\pi\nu t}\, dt \\ &= \int_{-\infty}^{0} e^{t-2i\pi\nu t}\, dt + \int_{0}^{\infty} e^{-t-2i\pi\nu t} \\ &= \frac{1}{1+2i\pi\nu} + \frac{1}{1-2i\pi\nu} \\ &= \frac{2}{1+4\pi^2\nu^2} \end{aligned} \tag{3.14}$$

1. The definition we will take here for the cardinal sine is $\mathrm{sinc}(x) = \frac{\sin x}{x}$. Another often used definition including the number π is : $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

2. More precisely Laplace's law is the function $p(t) = \frac{1}{2} e^{-|t|}$, whose integral is 1

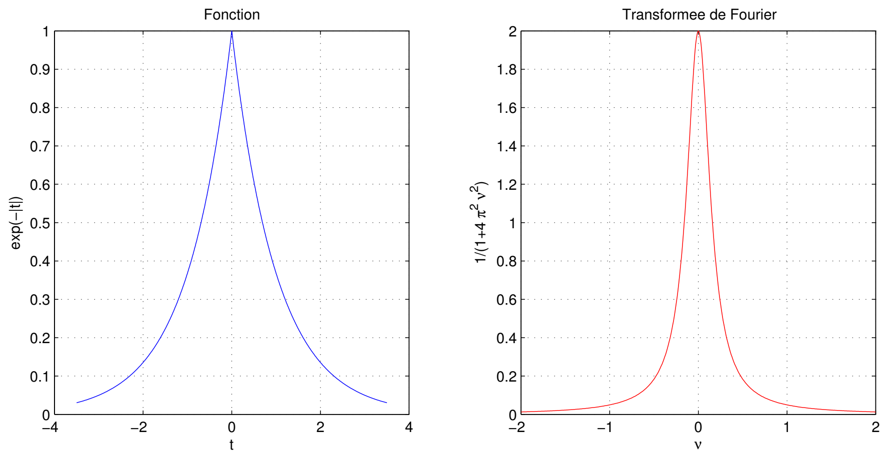


FIGURE 3.2 – Function $f(t) = e^{-|t|}$ (left) and its Fourier transform (right)

This function is a *Lorentzian* Its graph is represented in figure 3.2.

Dirac impulse

Let $f(t) = \delta(t)$. Its FT writes as

$$\begin{aligned} \hat{f}(\nu) &= \int_{-\infty}^{\infty} \delta(t)\, e^{-2i\pi\nu t}\, dt \\ &\downarrow \quad \text{use property } f(t)\delta(t) = f(0)\delta(t) \\ &= \int_{-\infty}^{\infty} \delta(t)\, dt \; = 1 \end{aligned} \tag{3.15}$$

We obtain the result that $\hat{f}(\nu)$ is 1 whatever ν, and we will write it : $\hat{f}(\nu) = \mathbf{1}(\nu)$. The converse is true as we will see in paragraph 3.2.4. We will retain that

$$\boxed{\begin{array}{l} f(t) = \delta(t) \iff \hat{f}(\nu) = \mathbf{1}(\nu) \\ f(t) = \mathbf{1}(t) \iff \hat{f}(\nu) = \delta(\nu) \end{array}} \tag{3.16}$$

3.2 Properties of the Fourier transform

3.2.1 Linearity

It is very easy to show that the Fourier transform is a linear operation, i.e. :
- For two functions f et g, we have $\mathcal{F}[f+g] = \mathcal{F}[f] + \mathcal{F}[g]$
- For a function f and a constant λ, we have $\mathcal{F}[\lambda f] = \lambda \mathcal{F}[f]$

3.2.2 Change of sign and conjugation

Change of sign : FT of $f(-t)$

It is written as :

$$\begin{aligned} \mathcal{F}[f(-t)] &= \int_{-\infty}^{\infty} f(-t)\, e^{-2i\pi\nu t}\, dt \\ &= \int_{-\infty}^{\infty} f(t)\, e^{+2i\pi\nu t}\, dt \quad = \quad \hat{f}(-\nu) \end{aligned} \tag{3.17}$$

Note that a reversal of the t axis (change $t \to -t$) results in a reversal of the ν axis :

$$\boxed{f(-t) \xrightarrow{\mathcal{F}} \hat{f}(-\nu)} \tag{3.18}$$

Conjugation : FT of $\overline{f(t)}$

It writes as :

$$\begin{aligned}\mathcal{F}\left[\overline{f(t)}\right] &= \int_{-\infty}^{\infty} \overline{f(t)}\, e^{-2i\pi\nu t}\, dt \\ &= \overline{\int_{-\infty}^{\infty} f(t)\, e^{+2i\pi\nu t}\, dt} = \overline{\hat{f}(-\nu)}\end{aligned} \tag{3.19}$$

3.2.3 Value at origin $\hat{f}(0)$

It writes as :

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(t)\, e^{-2i\pi 0 t}\, dt = \int_{-\infty}^{\infty} f(t)\, dt \tag{3.20}$$

Retain that :

the integral of a function is the value of its FT at 0

3.2.4 Change of scale

Consider a real constant $a \neq 0$ and a function f. We are interested in the FT of $f\left(\frac{t}{a}\right)$. It is written

$$\begin{aligned}\mathcal{F}\left[f\left(\frac{t}{a}\right)\right] &= \int_{-\infty}^{\infty} f\left(\frac{t}{a}\right)\, e^{-2i\pi\nu t}\, dt \\ &\downarrow \quad \text{variable change } y = t/a \text{ (caution } a \text{ may be negative)} \\ &= |a| \int_{-\infty}^{\infty} f(y)\, e^{-2i\pi y a\nu}\, dy = |a|\, \hat{f}(a\nu)\end{aligned} \tag{3.21}$$

Thus a dilation of the t axis (change $t \to t/a$) results in a compression of the ν axis (change $\nu \to a\nu$). This important property is illustrated in figure 3.3. We will retain that

$$\boxed{f\left(\frac{t}{a}\right) \xrightarrow{\mathcal{F}} |a|\, \hat{f}(a\nu)} \tag{3.22}$$

We can also remember the following sentence :

≪ *A function large in the direct space is narrow in the Fourier space.* ≫

Consequence : FT of $\mathbf{1}(t)$ Let the Gaussian function $g_\epsilon(t) = \exp -\pi(\epsilon t)^2$ with ϵ a positive real. Its FT is :

$$\hat{g}_\epsilon(\nu) = \frac{1}{\epsilon} \exp -\pi \left(\frac{\nu}{\epsilon}\right)^2 \tag{3.23}$$

When $\epsilon \to 0$, the function g_ϵ tends towards 1 while its FT $\hat{g}_0$ is a function whose width tends towards 0, height (value at the origin) becomes infinite and whose integral is 1 (it is easy to show that $\int_{-\infty}^{\infty} \hat{g}_\epsilon(\nu)\, d\nu = \int_{-\infty}^{\infty} \hat{g}_1(\nu)\, d\nu = \int_{-\infty}^{\infty} g_1(t)\, dt = \hat{g}_1(0) = 1$). We remark that $\hat{g}_0$ has the characteristics of a δ distribution as defined in paragraph 1.3.3, and can thus write that $\hat{g}_\epsilon(\nu)$ tends to $\delta(\nu)$ when $\epsilon \longrightarrow 0$. We will retain the following function/transform pair :

$$f(t) = \mathbf{1}(t) \iff \hat{f}(\nu) = \delta(\nu) \tag{3.24}$$

3.2.5 Translation

Consider a real constant a and a function f. We want to calculate the FT of $f(t+a)$:

$$\begin{aligned}\mathcal{F}\left[f\left(t+a\right)\right] &= \int_{-\infty}^{\infty} f\left(t+a\right)\, e^{-2i\pi\nu t}\, dt \\ &\downarrow \quad \text{variable change } y = t + a \\ &= \int_{-\infty}^{\infty} f(y)\, e^{-2i\pi(y-a)\nu}\, dy = e^{2i\pi a\nu}\, \hat{f}(\nu)\end{aligned} \tag{3.25}$$

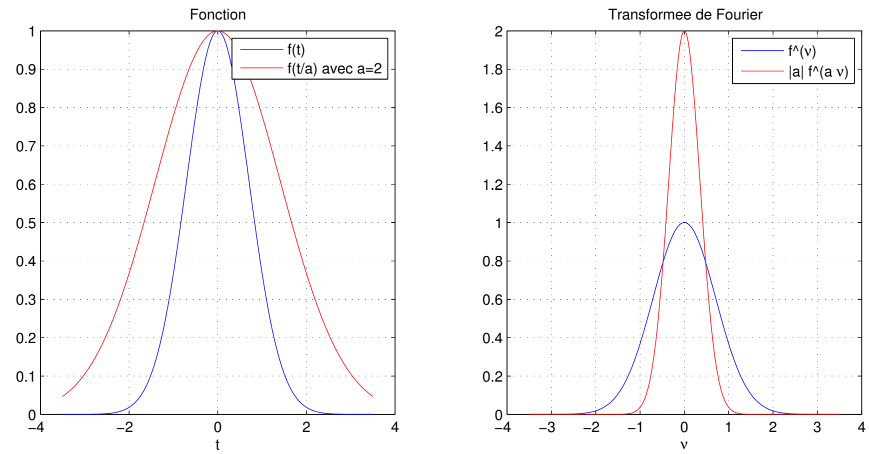


FIGURE 3.3 – Illustration of the scaling property. On the left a Gaussian function $f(t)$ (blue) and the same function $f(t/a)$ dilated by a factor $a = 2$ (red). On the right, the respective FTs : note the inversion of the proportions (the dilation in direct space has become a compression in Fourier space). We also note the value at the origin in Fourier space, higher for the red curve ($\hat{f}(0)$ is the integral of f).

Hence a translation of a along the t axis corresponds to a multiplication by a linear phase term $e^{2i\pi a\nu}$ in Fourier space. We will retain the property

$$f(t+a) \xrightarrow{\mathcal{F}} e^{2i\pi a\nu}\,\hat{f}(\nu) \tag{3.26}$$

Thus when performing a translation, the modulus of the Fourier transform is unchanged. Only the phase contains the information on this translation (addition of a linear contribution $\phi(u) = 2\pi ua$ to the phase of the TF of $f(t)$). Figure 3.4 shows an example in the case where $f(t)$ is a Gaussian.

3.2.6 Product by a phase term $e^{2i\pi\nu_0 t}$

Consider a real constant ν_0 and a function f. We are interested in the FT of $f(t)\,e^{2i\pi\nu_0 t}$. The calculation gives

$$\begin{aligned} \mathcal{F}\left[f(t)\,e^{2i\pi\nu_0 t}\right] &= \int_{-\infty}^{\infty} f(t)\,e^{-2i\pi(\nu-\nu_0)t}\,dt \\ &= \hat{f}(\nu-\nu_0) \end{aligned} \tag{3.27}$$

A multiplication by a linear phase term of frequency ν_0 in direct space results in a translation of $-\nu_0$ in Fourier space. This is the symmetrical property of that described in the previous paragraph (beware the - sign). We therefore retain these two properties :

$$\boxed{\begin{array}{lcl} f(t+a) & \xrightarrow{\mathcal{F}} & \hat{f}(\nu)\,e^{+2i\pi a\nu} \\ f(t)\,e^{+2i\pi\nu_0 t} & \xrightarrow{\mathcal{F}} & \hat{f}(\nu-\nu_0) \end{array}} \tag{3.28}$$

3.2.7 Inverse Fourier transform

The problem is the following : how to obtain $f(t)$ when we know $\hat{f}(\nu)$? It is the expression of the inverse FT that we are going to establish here. For this we go through a first step below :

Double Fourier transform

It will be denoted $\hat{\hat{f}}$ and is written as the FT of $\hat{f}$. The variables used will be : t for f, u for $\hat{f}$ and t_1 for $\hat{\hat{f}}$ (it should be noted that the dimension of t and t_1 is the same while that of u is the inverse of that of t and t_1). It comes

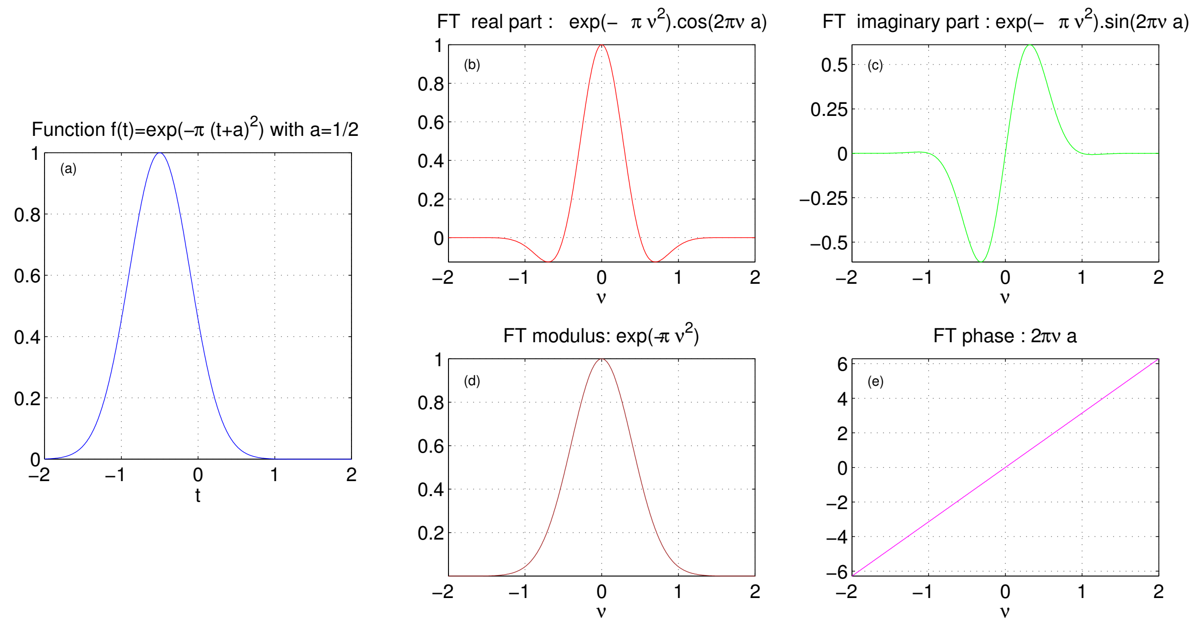


FIGURE 3.4 – Effect of a change of origin (translation) on the Fourier transform of a Gaussian. (a) : Gaussian function translated by a quantity $a = \frac{1}{2}$ in direct space. (b) and (c) : real and imaginary parts of the FT. (d) : module of the FT, independent of the shift a. (e) : phase of the FT, it is a straight line with slope $2\pi a$. The slope is positive if the function is shifted to the left, negative if the function is shifted to the right.

$$
\begin{aligned}
\hat{\hat{f}}(t_1) &= \mathcal{F}_{t_1}\left[\hat{f}(\nu)\right] \\
&= \int_{\nu=-\infty}^{\infty} \hat{f}(\nu)\, e^{-2i\pi t_1 \nu}\, d\nu \quad = \quad \int_{\nu=-\infty}^{\infty} \left[\int_{t=-\infty}^{\infty} f(t)\, e^{-2i\pi\nu t}\, dt\right] e^{-2i\pi t_1\nu}\, d\nu \\
& \qquad \downarrow \text{ swap integrals on } t \text{ and } \nu \\
&= \int_{t=-\infty}^{\infty} \underbrace{\left[\int_{\nu=-\infty}^{\infty} e^{-2i\pi\nu(t+t_1)}\, d\nu\right]}_{\text{FT of } \mathbf{1}(\nu) \text{ for the value } t+t_1} f(t)\, dt \\
&= \int_{t=-\infty}^{\infty} \delta(t+t_1)\, f(t)\, dt \underset{(f(t)\delta(t-a)=f(a)\delta(t-a))}{=} \int_{t=-\infty}^{\infty} \delta(t+t_1)\, f(-t_1)\, dt \\
&= f(-t_1)
\end{aligned} \tag{3.29}
$$

So the double Fourier transform gives the original function, with a sign change of the variable t (axis reversal). We will retain that :

$$
\boxed{\begin{aligned} \hat{\hat{f}}(t) &= f(-t) \\ f(t) &\xrightarrow{\mathcal{F}} \hat{f}(\nu) \xrightarrow{\mathcal{F}} f(-t) \end{aligned}} \tag{3.30}
$$

Inverse Fourier transform

We have just seen that the TF of $\hat{f}$ is $f(-t)$:

$$
\hat{f}(\nu) \xrightarrow{\mathcal{F}} f(-t) \tag{3.31}
$$

The Fourier integral relating here to the variable ν. We combine this result on the double Fourier transformation with the change of sign property seen in the paragraph 3.2.2. It comes

$$\hat{f}(-\nu) \xrightarrow{\mathcal{F}} f(t) \tag{3.32}$$

exand this relation :

$$\int_{-\infty}^{\infty} \hat{f}(-\nu)\, e^{-2i\pi\nu t}\, d\nu \quad = f(t) \tag{3.33}$$

The change of variable $\nu \rightarrow -\nu$ allows to write

$$\boxed{f(t) \;=\; \int_{-\infty}^{\infty} \hat{f}(\nu)\, e^{+2i\pi\nu t}\, d\nu \;=\; \mathcal{F}^{-1}[\hat{f}(\nu)]} \tag{3.34}$$

This equality links $\hat{f}(\nu)$ to $f(t)$: this is the expression of the inverse FT that we are looking for. It is very similar to that of the direct FT, the difference being the sign of the term $2i\pi\nu t$ in the exponential. We also see that

— For an even real function, inverse FT = FT
— For an odd real function, inverse FT = -FT

3.2.8 Physical significance of FT

This significance is found in the expression for the inverse FT, which allows us to write a function f as a continuous sum of terms $e^{+2i\pi\nu t}$ which we will name ≪ *harmonic components* ≫

$$f(t) \;=\; \int_{-\infty}^{\infty} \underbrace{\hat{f}(\nu)\, d\nu}_{\text{weighting factor}} \qquad \underbrace{e^{+2i\pi\nu t}}_{\text{harmonic components}} \tag{3.35}$$

These harmonic components $e^{+2i\pi\nu t}$ are trigonometric functions of frequency ν and are the complex equivalent of sinusoids. This is why we sometimes say that any function $f(t)$ (admitting a FT) can be written as a continuous sum of sinusoids of frequencies ν ranging from $-\infty$ to $+\infty$. The term $\hat{f}(\nu)$ represents the weight of the harmonic component of frequency ν in the expansion of f, and this is the physical meaning of the Fourier transform (we sometimes speak of *frequency spectrum* to denote FT). There are special cases that we will encounter a little later :

— The even real functions, which are written as a sum of cosines
— The even odd functions, which are written as a sum of sines
— Periodic functions, which are written as a discrete (and not continuous) sum of harmonic components
— Simple trigonometric functions like $\sin^2(t)$ which are written with a finite number of harmonic components

Example of a cosine

Let $f(t) = \cos(2\pi\nu_0 t)$. Its frequency is ν_0. This cosine is written as the sum of two complex exponentials

$$f(t) = \frac{1}{2}e^{2i\pi\nu_0 t} \;+\; \frac{1}{2}e^{-2i\pi\nu_0 t} \tag{3.36}$$

which in indeed an expansion into a sum of harmonic components analogous to the equation 3.35. There are only two harmonics in the cosine : that of frequency $+\nu_0$ and that of frequency $-\nu_0$. Each has an identical weight 1/2. This information is found in the Fourier transform of f. To calculate it, we use the result of the paragraph 3.2.6)

$$g(t)\, e^{2i\pi\nu_0 t} \quad \xrightarrow{\mathcal{F}} \quad \hat{g}(\nu - \nu_0)$$

avec $g(t) = \mathbf{1}(t)$. It comes :

$$\hat{f}(\nu) = \frac{1}{2}\delta(\nu - \nu_0) \;+\; \frac{1}{2}\delta(\nu + \nu_0) \tag{3.37}$$

The FT of a cosine of frequency ν_0 is the sum of two δ distributions, centered at $\pm\nu_0$. It is zero for any other value of ν : we deduce that the cosine contains no frequency other than $\pm\nu_0$ in its Fourier expansion. It is even : this is why the two frequencies $\pm\nu_0$ have the same weight ($\frac{1}{2}$) in the Fourier expansion of f.

Case of a real and even function

Let f be a real and even function, admitting an FT $\hat{f}$ which is also real and even. It is easy to show that the equation 3.34 can be written

$$f(t) \; = 2 \int_0^{\infty} \hat{f}(\nu) \; \cos(2\pi\nu t) \; d\nu \tag{3.38}$$

A real and even function is thus written as a continuous sum of cosines at all frequencies ν between 0 and ∞, weighted by $\hat{f}(\nu)$. In other words, by adding a large number (ideally an infinity) of cosines with the right weights, one can construct any function (provided it admits a FT).

Example of a rectangle function $f(t) = \Pi(t)$. Its FT writes $\hat{f}(\nu) = \text{sinc}(\pi\nu)$. The equation 3.38 allows to write

$$\Pi(t) \; = \; 2 \int_0^{\infty} \; \text{sinc}(\pi\nu) \; \cos(2\pi\nu t) \; d\nu \tag{3.39}$$

By means of a computer, it is easy to approximate this integral by Riemann sum

$$\Pi(t) \; \simeq \; 2 \sum_{n=0}^{M} \text{sinc}(\pi\nu_n) \; \cos(2\pi\nu_n t) \; \delta\nu \tag{3.40}$$

with $\delta\nu$ a frequency step and $\nu_n = n\delta\nu$ the sampled values of the frequency ν. M is a large number, ideally infinite for the Riemann series, but if one wants to make a numerical calculation it is necessary to fix a limit value.
The figure 3.5 shows how the rectangle function is constructed when we sum a few terms of the series. In particular the graph (c_4), obtained as a sum of only 4 terms ($\nu_n = 0, 0.5, 1.5, 2.5$) already evokes the form of the rectangle function. Another example is shown in figure 3.6 with a much higher number of terms (up to $M = 10000$) and a tighter $\delta\nu$ frequency sampling. It is interesting to observe the convergence towards the rectangle function as M increases. It can be seen that the flat parts are fairly quickly reconstructed while oscillations are observed in the vicinity of the discontinuities (Gibbs phenomenon). These oscillations disappear when $M \to \infty$.

3.2.9 Derivation

Consider a function f admitting a FT (and therefore integrable on $\mathbb{R}$). It is written as the inverse FT of $\hat{f}$, i.e.

$$\begin{aligned}
f(t) \quad &= \quad \int_{\nu=-\infty}^{\infty} \hat{f}(\nu)\, e^{2i\pi\nu t} \, d\nu \\
&\downarrow \quad \text{derive with respect to } t \\
\frac{d}{dt} f(t) \quad &= \quad \frac{d}{dt} \int_{\nu=-\infty}^{\infty} \hat{f}(\nu)\, e^{2i\pi\nu t} \, d\nu \\
&\downarrow \quad \text{derive under the integral sign } \textstyle\int \text{ (Leibniz rule)} \\
&= \quad \int_{\nu=-\infty}^{\infty} \hat{f}(\nu) \, \frac{d}{dt} \, e^{2i\pi\nu t} \, d\nu \\
&= \quad \int_{\nu=-\infty}^{\infty} \hat{f}(\nu)\, 2i\pi\nu \, e^{2i\pi\nu t} \, d\nu \\
&= \quad \mathcal{F}^{-1} \left[2i\pi\nu \hat{f}(\nu) \right]
\end{aligned} \tag{3.41}$$

We will retain the propoerty :

$$\boxed{f'(t) \; \xrightarrow{\mathcal{F}} \; 2i\pi\nu \hat{f}(\nu)} \tag{3.42}$$

This translates into the idea that a derivative corresponds to an increase in the weight of high frequencies (large values of $|\nu|$). We talk about *high pass filtering* and we will develop this idea in the chapter dealing with filtering. From the previous equation we can deduce the result below, sometimes useful for certain calculations :

$$t \,.\, f(t) \; \xrightarrow{\mathcal{F}} \; -\frac{1}{2i\pi} \frac{d}{d\nu} [\hat{f}(\nu)] \tag{3.43}$$

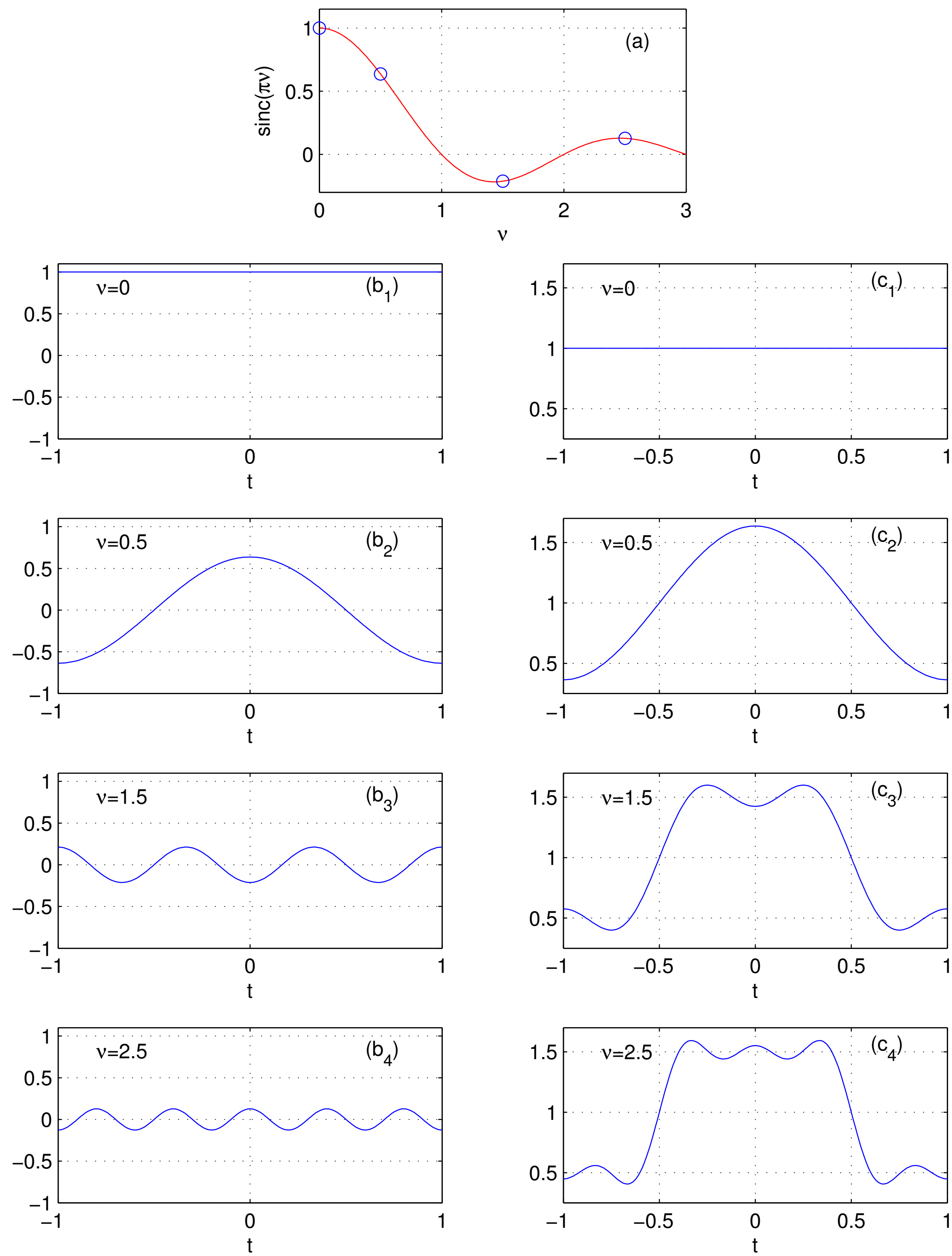


FIGURE 3.5 – Approximation of a rectangle function $f(t) = \Pi(t)$ as sum of cosines (equation 3.40). (a) : FT of the rectangle function $\mathrm{sinc}(\pi\nu)$. The circles correspond to the values of $\nu = 0, 0.5, 1.5, 2.5$ used in the sum. (b_1) to (b_4) : term $\mathrm{sinc}(\pi\nu)\cos(2\pi\nu t)$ for $\nu = 0, 0.5, 1.5, 2.5$. (c_1) to (c_4) : sum of the terms corresponding to $\mathrm{sinc}(\pi\nu)\cos(2\pi\nu t)$ for $\nu = 0$, $\nu = 0, 0.5$, $\nu = 0, 0.5, 1.5$, and $\nu = 0, 0.5, 1.5, 2.5$.

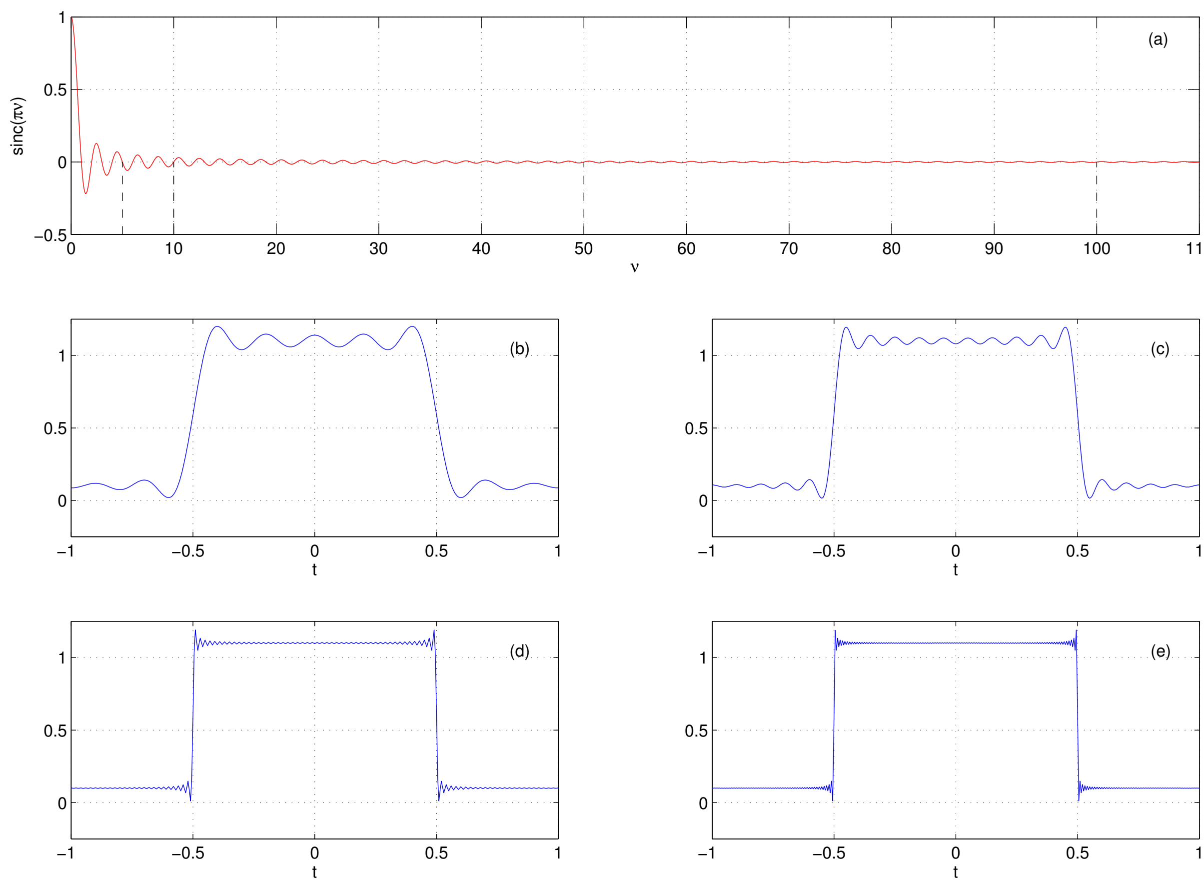


Figure 3.6 – Approximation of a rectangle function $f(t) = \Pi(t)$ by the sum of the series (eq. 3.40) with $\delta\nu = 0.01$. (a) : FT of the rectangle function $\mathrm{sinc}(\pi\nu)$. (b) to (e) : calculation of the series for 4 maximum values ν_M of ν corresponding to $\nu_M = 5, 10, 50, 100$ (i.e. an upper bound $M = 500, 1000, 5000, 10000$ for n). The vertical lines of the graph (a) correspond to the 4 values of ν_M.

Similarly, the FT of the primitive of f is written

$$\boxed{\int f(t)\,dt \xrightarrow{\mathcal{F}} \frac{1}{2i\pi\nu}\hat{f}(\nu)} \tag{3.44}$$

which corresponds to an attenuation of high frequencies (low-pass filtering).

3.2.10 FT of a convolution and a product

Let f and g be two functions of the same variable t, and h their convolution product

$$h(t) \;=\; [f * g](t) \;=\; \int_{t'=-\infty}^{\infty} f(t')\,g(t-t')\,dt' \tag{3.45}$$

The FT of h writes

$$\begin{aligned}
\hat{h}(\nu) &= \int_{t=-\infty}^{\infty} h(t)\,e^{-2i\pi\nu t}\,dt \\
&= \int_{t=-\infty}^{\infty}\int_{t'=-\infty}^{\infty} f(t')\,g(t-t')\,dt'\,e^{-2i\pi\nu t}\,dt \\
&\downarrow \;\text{ swap of integrals} \\
&= \int_{t'=-\infty}^{\infty} f(t')\underbrace{\int_{t=-\infty}^{\infty} g(t-t')\,e^{-2i\pi\nu t}\,dt}_{\text{FT of } g(t-t')}\,dt' \\
&= \hat{g}(\nu)\int_{t'=-\infty}^{\infty} f(t')\,e^{-2i\pi\nu t'}\,dt' \\
&= \hat{g}(\nu)\,.\,\hat{f}(\nu)
\end{aligned} \tag{3.46}$$

We have shown the well-known result that the FT of a convolution is a simple product of individual transforms. Let us now show the converse property, i.e. the FT of a product of functions is the convolution of the transforms :

$$\begin{aligned}
\mathcal{F}[f(t).g(t)] &= \int_{t=-\infty}^{\infty} f(t)\,g(t)\,e^{-2i\pi\nu t}\,dt \\
&\downarrow \;\text{ write } g(t) \text{ as inverse FT of } \hat{g}(\nu') \\
&= \int_{t=-\infty}^{\infty} f(t)\int_{\nu'=-\infty}^{\infty}\hat{g}(\nu')\,e^{2i\pi\nu' t}\,d\nu'\,e^{-2i\pi\nu t}\,dt \\
&= \int_{\nu'=-\infty}^{\infty}\hat{g}(\nu')\int_{t=-\infty}^{\infty} f(t)\,e^{-2i\pi(\nu-\nu')t}\,dt\,d\nu' \\
&= \hat{g}(\nu')\,\hat{f}(\nu-\nu')dt' \\
&= [\hat{g} * \hat{f}](\nu)
\end{aligned} \tag{3.47}$$

We will retain the following properties :

$$\boxed{\begin{array}{lcl}
f(t).g(t) & \xrightarrow{\mathcal{F}} & \left[\hat{f} * \hat{g}\right](\nu) \\
\left[f * g\right](t) & \xrightarrow{\mathcal{F}} & \hat{f}(\nu)\,.\,\hat{g}(\nu)
\end{array}} \tag{3.48}$$

3.2.11 FT of causal function

A causal function $f(t)$ has the property of being zero for $t < 0$, and satisfies $f(t) = f(t).H(t)$ with H the Heaviside function.

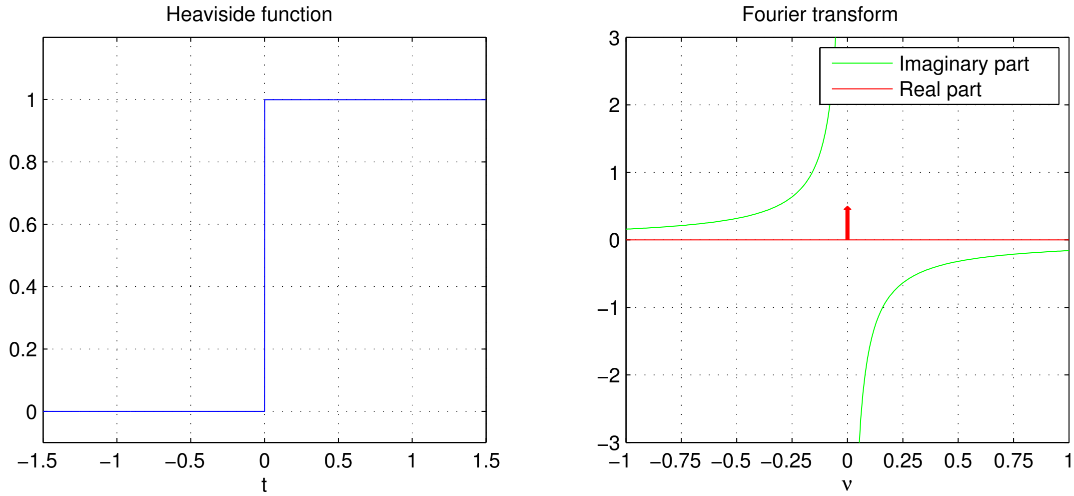


FIGURE 3.7 – Heaviside unit step (left) and its Fourier transform (right).

FT of $H(t)$

We use the relation 3.44 (FT of the primitive of a function) that we apply to $f(t) = \delta(t)$. The antiderivative of δ is the Heaviside step multiplied by an integration constant. We will choose here the constant equal to $-\frac{1}{2}$ in order to obtain the following odd primitive :

$$f(t) = \delta(t) \quad \xrightarrow{\text{primitive}} \quad F(t) = H(t) - \frac{1}{2} \tag{3.49}$$

and since F is odd it verifies

$$\int_{-\infty}^{\infty} F(t)\, dt = \hat{F}(0) = 0 \tag{3.50}$$

We now apply the relation 3.44 valid for $\nu \neq 0$

$$\hat{F}(\nu) = \frac{1}{2i\pi\nu}\hat{f}(\nu) \;=\; \frac{1}{2i\pi\nu} \tag{3.51}$$

So :

$$\begin{cases} \hat{F}(\nu) = \frac{1}{2i\pi\nu} & \text{for } \nu \neq 0 \\ \hat{F}(\nu) = 0 & \text{for } \nu = 0 \end{cases} \tag{3.52}$$

This is the function $\frac{1}{2i\pi\nu}$ deprived of its singularity at 0, we will denote it

$$\hat{F}(\nu) = \text{vp}\left[\frac{1}{2i\pi\nu}\right] \tag{3.53}$$

the term vp for *principal value*. It therefore comes, since $H(t) = F(t) + \frac{1}{2}$:

$$\boxed{\hat{H}(\nu) = \frac{1}{2}\delta(\nu) \;+\; \text{vp}\left[\frac{1}{2i\pi\nu}\right]} \tag{3.54}$$

It is the Heaviside step FT, which has an even real part and an odd imaginary part (general property of FTs of real functions). Its graph is represented in figure 3.7. We can deduce the FT from the sign function defined by

$$\text{sgn}(t) = \begin{matrix} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{matrix} \quad \xrightarrow{\mathcal{F}} \quad \hat{\text{sgn}}(\nu) = \text{vp}\left[\frac{1}{i\pi\nu}\right] \tag{3.55}$$

FT of a causal function — Hilbert transform

Let f be a causal function satisfying $f(t) = f(t).H(t)$. We have :

$$\begin{aligned}
\hat{f}(\nu) &= \mathcal{F}[f(t).H(t)] = \hat{f}(\nu) \, * \, \hat{H}(\nu) \\
&= \hat{f}(\nu) \, * \, \left(\frac{1}{2}\delta(\nu) \, + \, \mathrm{vp}\left[\frac{1}{2i\pi\nu}\right] \right) \\
&= \frac{1}{2}\hat{f}(\nu) \, + \hat{f}(\nu) \, * \, \mathrm{vp}\left[\frac{1}{2i\pi\nu}\right] \\
&= \frac{1}{2}\hat{f}(\nu) \, - \frac{i}{2} \, \mathrm{vp}\left[\hat{f}(\nu) \, * \, \frac{1}{\pi\nu}\right]
\end{aligned} \tag{3.56}$$

The principal value of the convolution integral of $f(\nu)$ by $\frac{1}{\pi\nu}$ is the limit $\epsilon \to 0$ of

$$\mathrm{vp}\left[\hat{f}(\nu) \, * \, \frac{1}{\pi\nu}\right] = \int_{\infty}^{-\epsilon} \frac{\hat{f}(\nu - \nu')}{\pi\nu'} \, d\nu' \quad + \quad \int_{\epsilon}^{-\infty} \frac{\hat{f}(\nu - \nu')}{\pi\nu'} \, d\nu' \tag{3.57}$$

In the following we will omit the explicit writing of the principal value and assume an extension by continuity in 0. It comes :

$$\hat{f}(\nu) = -i \, \hat{f}(\nu) * \frac{1}{\pi\nu} \tag{3.58}$$

which is the equivalent in Fourier space of the relation $f(t) = f(t).H(t)$. The quantity $\hat{f}(\nu) * \frac{1}{\pi\nu}$ is called *Hilbert transform* of $\hat{f}$.

3.2.12 How to calculate a FT

The FT is defined by an integral (eq. 3.2) but it is generally not necessary to calculate it. For example let the function

$$f(t) = \cos(3t) \, \exp -\pi(t - t_0)^2 \tag{3.59}$$

with t_0 real. f is therefore the product of a function $f_1(t) = \cos(3t)$ by a Gaussian $f_2(t) = \exp -\pi(t - t_0)^2$. To calculate its TF, we use the property 3.48 to write

$$\hat{f}(\nu) = \hat{f}_1(\nu) \, * \, \hat{f}_2(\nu) \tag{3.60}$$

To calculate $\hat{f}_1(\nu)$, we decompose the sine into the sum of two exponentials :

$$f_1(t) = \frac{1}{2}e^{3it} + \frac{1}{2}e^{-3it} \tag{3.61}$$

and we use the property 3.27 to deduce

$$\hat{f}_1(\nu) = \frac{1}{2} \, \delta\left(\nu + \frac{3}{2\pi}\right) + \frac{1}{2} \, \delta\left(\nu - \frac{3}{2\pi}\right) \tag{3.62}$$

To calculate $\hat{f}_2(\nu)$ we combine the translation property 4.9 with the FT of the Gaussian (eq. 3.13). It comes

$$\hat{f}_2(\nu) = e^{-\pi\nu^2} \, e^{-2i\pi\nu t_0} \tag{3.63}$$

And finally, we apply the property 2.4 $f(x) * \delta(x - a) = f(x - a)$ to obtain

$$\hat{f}(\nu) = \frac{1}{2} \, e^{-\pi\left(\nu - \frac{3}{2\pi}\right)^2} \, e^{-2i\pi\left(\nu - \frac{3}{2\pi}\right)t_0} \, + \, \frac{1}{2} \, e^{-\pi\left(\nu + \frac{3}{2\pi}\right)^2} \, e^{-2i\pi\left(\nu + \frac{3}{2\pi}\right)t_0} \tag{3.64}$$

This example illustrates the importance of having a form grouping the known FTs and the essential properties. This form is given in the appendix.

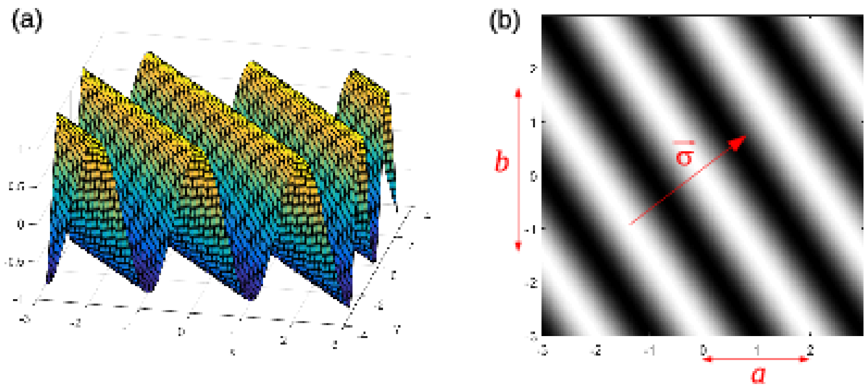


FIGURE 3.8 – Representation of a 2D sinudoidal function $f(x,y) = \cos(2\pi\vec{\sigma}.\vec{\rho}) = \cos(2\pi(ux+vy))$. (a) : perpective plot. (b) : grayscale plot. The frequency vector $\vec{\sigma} = (u,v)$ has been drawn on the right plot ; it is perpendicular to the ridge lines of the function, its modulus is the frequency of the oscillations measured along the unit vector $\hat{\sigma}$. Its components are $u = \frac{1}{a}, v = \frac{1}{b}$ with a and b the periods in the x and y directions.

3.3 Two dimensionnal Fourier transform

3.3.1 Definition

The 2D Fourier transform of a function of two variables $f(x,y)$ is defined as

$$\hat{f}(u,v) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-2i\pi(ux+vy)}\, dx\, dy \tag{3.65}$$

The variables u and v are spatial frequencies associated to the space variables x and y. They define, in the (u,v) plane, a "spatial frequency vector" $\vec{\sigma} = \left|\begin{array}{l} u \\ v \end{array}\right.$ (see Fig. 3.8). As for the 1D Fourier transform, the idea is that a function $f(x,y)$ can be expressed as a sum of 2D complex sinusoids of any period and any orientation.
The 2D inverse Fourier transform is

$$f(x,y) = \iint_{-\infty}^{\infty} \hat{f}(u,v)\, e^{+2i\pi(ux+vy)}\, dx\, dy \tag{3.66}$$

3.3.2 Specific properties for 2D Fourier transform

Separable functions : if a function $h(x,y)$ is the product of two functions of one variable $f(x)$ and $g(y)$, then its 2D Fourier transform is also a separable function, i.e.

$$h(x,y) \;=\; f(x)\,.\,g(y) \quad \xrightarrow{\mathcal{F}} \quad \hat{h}(u,v) \;=\; \hat{f}(u)\,.\,\hat{g}(v) \tag{3.67}$$

This property must not to be confused with the Fourier transform of a product of functions of the same variables (Eq. 3.48) : here the variables for f and g are different, and the 2D transform is a double integral.

Radial functions of the type $f(x,y) = f(\rho)$ **with** $\rho = \sqrt{x^2+y^2}$: the 2D Fourier transform $\hat{f}(u,v)$ is also a radial function $F(q)$ with $q = \sqrt{u^2+v^2}$. It takes the following form known as *Hankel transform* :

$$F(q) = \hat{f}(u,v) = \int_0^{\infty} 2\pi\rho\, f(\rho)\, J_0(2\pi q\rho)\, d\rho \tag{3.68}$$

where $J_0(x)$ is the zero order Bessel function. The Hankel transform $f(\rho) \longrightarrow F(q)$ is not to be confused with the 1D Fourier transform (Eq. 3.2).

3.4 Transfer function and filtering

3.4.1 Example : RC circuit

We consider the RC circuit of the figure 3.9. A generator delivers an alternating voltage $x(t)$ in a circuit made up of a resistor R and a capacitor C connected in series. The voltage $y(t)$ is measured across the terminals of the

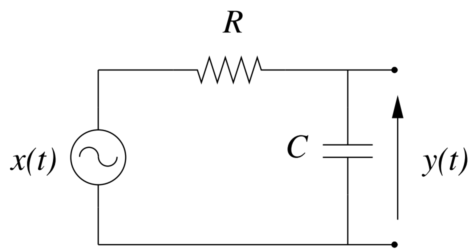


FIGURE 3.9 – RC circuit. $x(t)$ is the input voltage, $y(t)$ is the voltage measured across the capacitor.

capacitor. Using the vocabulary presented in paragraph 2.3.2, $x(t)$ will be called the "input signal" and $y(t)$ the "output signal".
The voltage law makes it possible to write the differential equation to which the charge $q(t)$ of the capacitor satisfies :

$$R\,\frac{dq}{dt} + \frac{q}{C} = x(t) \tag{3.69}$$

let $y = \frac{q}{C}$, it comes :

$$RC\,\frac{dy}{dt} + y = x(t) \tag{3.70}$$

It is a linear differential equation whose solution $y(t)$ is written as the convolution

$$y(t) = x(t) \,*\, R(t) \tag{3.71}$$

with $R(t)$ the impulse response, as we saw in paragraph 2.3.1. Another way to show this is to calculate the FT of the differential equation :

$$RC\;2i\pi\nu\,\hat{y}(\nu) + \hat{y}(\nu) = \hat{x}(\nu) \tag{3.72}$$

The Fourier transformation made it possible to transform the differential equation into a linear equation. the calculation of $\hat{y}(\nu)$ is immediate :

$$\hat{y}(\nu) = \hat{x}(\nu)\;.\;\frac{1}{1 + 2i\pi\nu\;RC} \tag{3.73}$$

and setting

$$\hat{R}(\nu) = \frac{1}{1 + 2i\pi\nu\;RC} \tag{3.74}$$

we make $\hat{y}$ appear as the product of two quantities

$$\hat{y}(\nu) = \hat{x}(\nu)\;.\hat{R}(\nu) \tag{3.75}$$

This relation between the input and output signals in the Fourier space is called *linear filtering relation*. The output signal $y(t)$ is thus also named *filtered signal*. By inverse FT we find the equation 3.71. We can also calculate $R(t)$ [3] :

$$R(t) = \frac{1}{RC}\;H(t)\;\exp -\frac{t}{RC} \tag{3.76}$$

The quantity $\hat{R}(\nu)$ is called *transfer function*. This is the FT of the impulse response. And like the impulse response, it only depends on the characteristics of the RC circuit (resistance R and capacitance C) and not on the input voltage $x(t)$. Its graph (real/imaginary parts, modulus and phase) is represented in figure 3.10.
Another representation in logarithmic scale (for $\nu > 0$) is shown in figure 3.11. This representation widely used in the field of electronics is known as *Bode diagram*. The modulus of $\hat{R}(\nu)$ is converted into decibels (dB) by the formula

$$G(\nu) = 20\;\log_{10}(|\hat{R}(\nu)|) \tag{3.77}$$

so that a drop of a factor 10 of $|\hat{R}(\nu)|$ results in a loss of 20 dB. $G(\nu)$ is sometimes called "gain". This representation is convenient because it shows two different regimes for the behavior of G :

- $G(\nu)$ is constant for $|\nu| \ll \frac{1}{RC}$, its value is 0
- $G(\nu)$ is a straight line with negative slope (often expressed in decibels per decade) for $|\nu| \gg \frac{1}{RC}$

3. The calculation is easy if we have previously calculated the FT of $H(t)\;\exp(-t)$ with the Fourier integral.

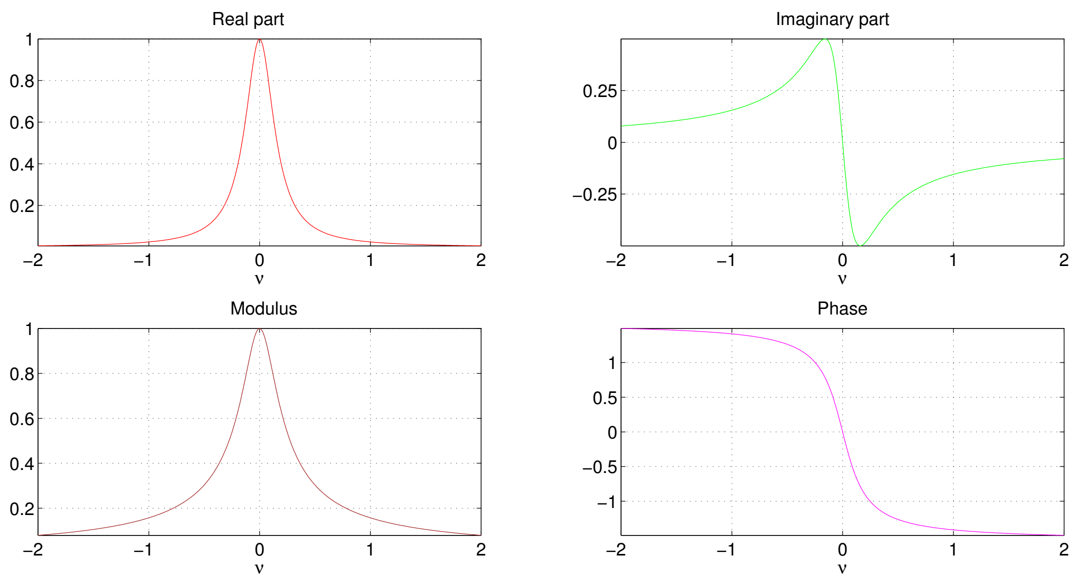


FIGURE 3.10 – Transfer function of the RC circuit for $RC = 1$.

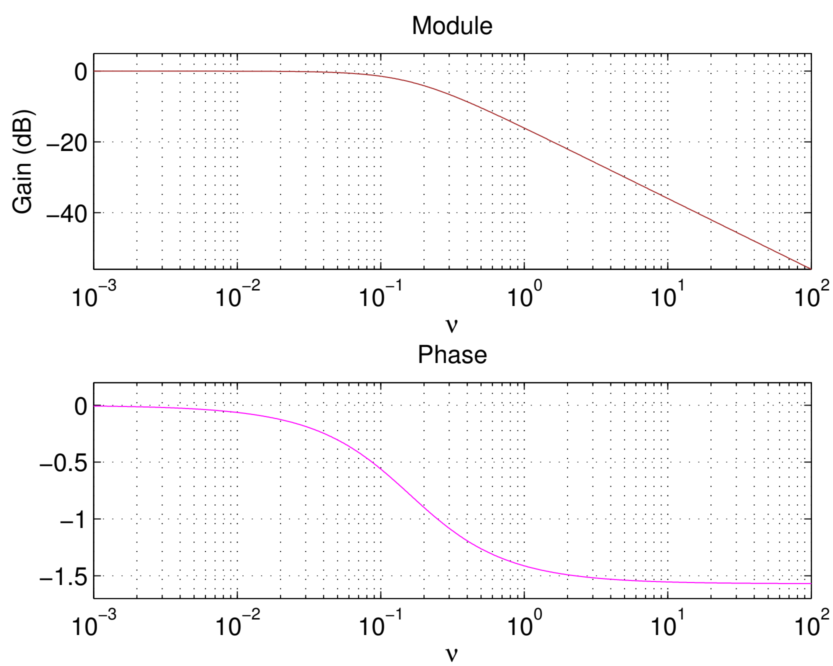


FIGURE 3.11 – Bode diagram of the transfer function $\hat{R}(\nu)$ of the RC circuit for $RC = 1$. Top : gain $G(\nu) = 20\log_{10}(|\hat{R}(\nu)|)$ as a function of ν. Bottom : phase of $\hat{R}(\nu)$ as a function of ν.

Physical significance of the transfer function

The physical significance of this transfer function is easy to understand when the input voltage $x(t)$ is sinusoidal with frequency ν_0, for example

$$x(t) = x_0 \cos(2\pi\nu_0 t) \tag{3.78}$$

In this case $\hat{x}(\nu)$ is the sum of two δ distributions :

$$\hat{x}(\nu) = \frac{x_0}{2}[\delta(\nu - \nu_0) + \delta(\nu + \nu_0)] \tag{3.79}$$

The filtering corresponding to the equation 3.75 allows to write $\hat{y}(\nu)$ as a sum of two distributions δ

$$\hat{y}(\nu) = \frac{x_0}{2}[\hat{R}(\nu_0)\,\delta(\nu - \nu_0) + \hat{R}(-\nu_0)\delta(\nu + \nu_0)] \tag{3.80}$$

and we obtain the output signal $y(t)$ by inverse FT, using the property $\hat{R}(-\nu_0) = \overline{\hat{R}(\nu_0)}$

$$y(t) = x_0\,|\hat{R}(\nu_0)| \cos(2\pi\nu_0 t + \phi_0) \tag{3.81}$$

with ϕ_0 the phase of $\hat{R}(\nu_0)$. Retain that :

When the input signal is a sinusoid of frequency ν_0, the output signal is also a sinusoid,
- with the same frequency ν_0,
- whose amplitude has been multiplied by $|\hat{R}(\nu_0)|$,
- which is out of phase by an amount $\phi_0 = \arg\left[\hat{R}(\nu_0)\right]$

The role of the modulus transfer function is comparable to that of a *equalizer* in a stereo system : it acts as an attenuation coefficient for the signal at the frequency ν_0. The phase of the transfer function acts for its part by a phase shift of this signal. The Bode diagram in figure 3.11 shows that a low frequency signal $\nu \ll \frac{1}{RC}$ will be almost unchanged ($y(t) \simeq x(t)$) , the RC circuit has no effect on low frequency signals. On the other hand at high frequency $\nu \gg \frac{1}{RC}$, the output signal will have a low amplitude, which tends towards 0 as ν increases. Such filtering is therefore called ≪ low-pass ≫ because it "passes through" low frequencies and blocks high frequencies. See illustration in figure 3.12.

3.4.2 Some definitions

Stationary linear systems

We consider a physical system which links an input signal x to an output signal y by an "input-output" relation. For example :
- The RC circuit of the previous paragraph : x is the generator voltage, y the voltage measured across the capacitor, the input-output relationship is a convolution
- The spring of paragraph 2.3.1 : x is the force exerted on the spring, y is its elongation. The input-output relationship is also a convolution
- A microphone that transforms a sound signal (x is the pressure exerted on the membrane of the microphone) into an electrical signal y
- A camera which forms the image $I(x', y')$ of a light source of intensity $I_0(x', y')$ at a point of coordinates x', y' in the plane. The relation between I_0 and I is an input-output relation, between two functions of two variables.

A common point of the previous examples is that they make it possible to carry out a measurement of the input signal x which is sometimes unknown (in the example of the microphone the air pressure on the membrane is unknown). The measured quantity is y. The relation between x and y depends on the physics of the system (in the case of the spring, the fundamental relation of dynamics makes it possible to obtain it). This relation takes the form of a convolution in the case of particular systems called *linear and stationary.*
A system is said to be **linear** if its response to a linear combination of input signals is the linear combination of output signals :

$$\text{Input signal : } a_1\,x_1 + a_2\,x_2 \quad \Longrightarrow \quad \text{Output signal : } a_1\,y_1 + a_2\,y_2 \tag{3.82}$$

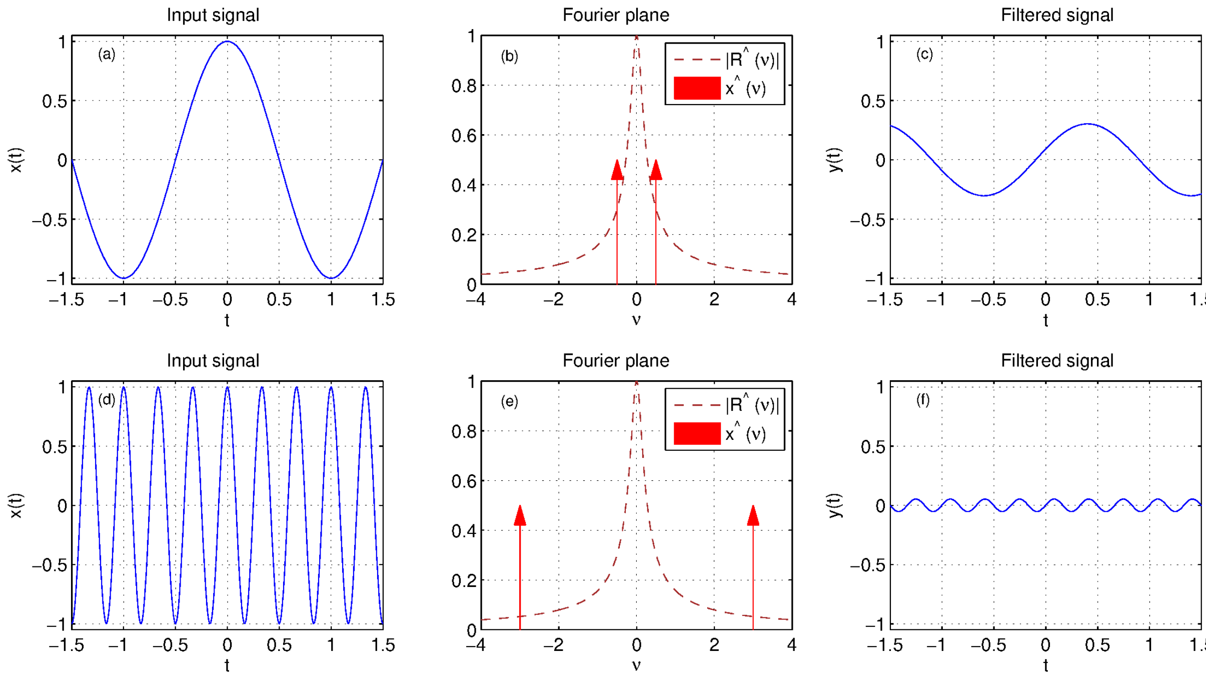


FIGURE 3.12 – Illustration of the effect of filtering a sinusoidal signal $x(t)$ by an RC circuit (with $RC = 1$). (a) input signal $x(t) = \cos(2\pi\nu_0 t)$ with $\nu_0 = 0.5$. (b) representations in Fourier space of the transfer function (module) and of $\hat{x}(\nu)$ (sum of two Diracs). (c) filtered signal $y(t)$ which is a sinusoid of frequency ν_0, damped and out of phase. (d), (e), (f) : same thing with $\nu_0 = 3$ There is a greater attenuation for this higher frequency, as well as a phase shift of almost $\pi/2$ (as predicted by the Bode diagram).

with y_1 (resp. y_2) the response to the input signal x_1 (resp. x_2), and a_1 and a_2 constants.

A system is said to be **stationary** if its characteristics are invariant under translation, whether in time (in the case of time-dependent input and output signals) or in space (in the case of signals depending on space variables like intensity in an image). In this case a translation of the input signal results in an identical translation of the output signal :

$$\begin{array}{lll} \text{Input signal : } x(t) & \Longrightarrow & \text{Output signal : } y(t) \\ \text{Input signal : } x(t-t_0) & \Longrightarrow & \text{Output signal : } y(t-t_0) \end{array} \tag{3.83}$$

with t the variable on which the input and output signals depend and t_0 the translation.

It is easy to show that the input-output relation of such a system is a convolution by an impulse response $R(t)$. Indeed $x(t)$ can be written as a continuous sum of δ distributions

$$x(t) \;=\; x(t) * \delta(t) \;=\; \int_{-\infty}^{\infty} x(t')\,\delta(t-t')\,dt' \tag{3.84}$$

The response of the system to an input signal $\delta(t-t')$ is $R(t-t')$ if the system is stationary. And since the system is also linear, its response to a sum of distributions δ is written

$$\text{Input signal : } x_0\delta(t-t_0)+x_1\delta(t-t_1)+x_2\delta(t-t_2)\ldots \quad \Longrightarrow \quad \text{Output signal : } y_0R(t-t_0)+y_1R(t-t_1)+x_2R(t-t_2)\ldots \tag{3.85}$$

or,

$$\text{Input signal : } \sum_n x_n\delta(t-t_n) \quad \Longrightarrow \quad \text{Output signal : } \sum_n x_n R(t-t_n) \tag{3.86}$$

or again, going to the continuous form

$$\text{Input signal : } \int_{-\infty}^{\infty} x(t')\,dt'\;\delta(t-t') \quad \Longrightarrow \quad \text{Output signal : } \int_{-\infty}^{\infty} x(t')\,dt'\;R(t-t') \tag{3.87}$$

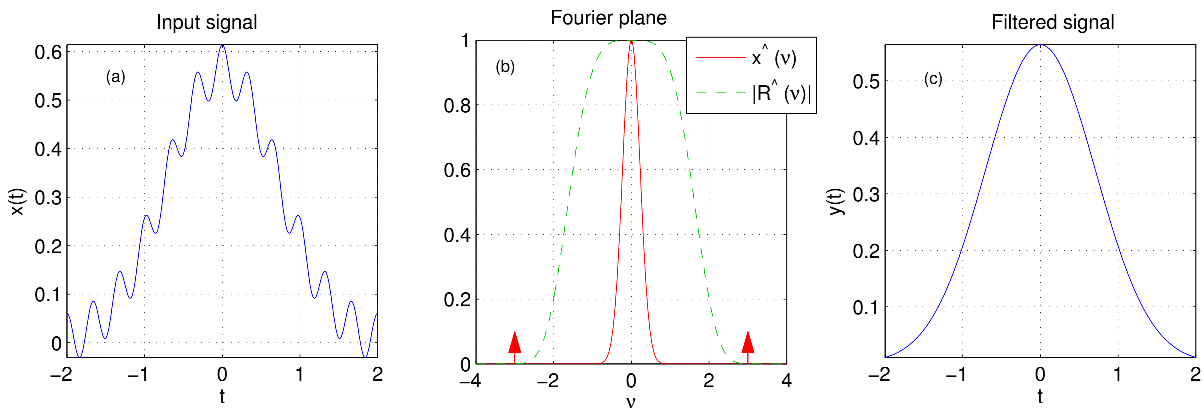


FIGURE 3.13 – Low-pass filtering of a signal. (a) input signal $x(t) = a_1 \exp(-t^2) + a_0 \cos(2\pi\nu_0 t)$ with $\nu_0 = 3$, $a_1 = \frac{1}{\sqrt{\pi}}$ and $a_0 = 0.05$. (b) representations in Fourier space of $\hat{x}(\nu)$ (sum of a Gaussian centered at $\nu = 0$ and of two Diracs coming from the term $a_0 \cos(2\pi\nu_0 t)$). The transfer function is shown in dotted lines. (c) filtered signal $y(t)$ in which the cosine term has almost completely disappeared.

Thus, for a stationary linear system :

$$\boxed{\text{Input signal : } x(t) \quad \Longrightarrow \quad \text{Output signal : } y(t) = x(t) * R(t)} \tag{3.88}$$

low pass filtering

We consider a stationary linear impulse response system $R(t)$, which connects an input signal $x(t)$ to an output signal $y(t)$. The relation between $\hat{x}$ and $\hat{y}$ in Fourier space is written

$$\hat{y}(\nu) = \hat{R}(\nu)\,\hat{x}(\nu) \tag{3.89}$$

We say that this filtering is of the *low-pass* type if the transfer function (its modulus) tends towards 0 in the distance, i.e.

$$|\hat{R}(\nu)| \xrightarrow[|\nu|\longrightarrow\infty]{} 0 \tag{3.90}$$

This concretely means that the high frequency harmonic components in the signal $x(t)$ are attenuated by this type of filtering. The figure 3.13 shows an example of low pass filtering on a signal $x(t)$ composed of a Gaussian and a cosine : $x(t) = a_1 \exp(-t^2) + a_0 \; cos(2\pi\nu_0 t)$ with constants $a_1 = \frac{1}{\sqrt{\pi}}$ and $a_0 = 0.05$. The cosine frequency is $\nu_0 = 3$. The transfer function is $\hat{R}(\nu) = \exp\left(-\frac{\nu^4}{10}\right)$. It is equal to 1 near the origin and decreases very strongly when $|\nu| > 2.6$ (we have $R(2.6) \simeq 0.01$).
In Fourier space, $\hat{x}(\nu)$ is the sum of three terms : a Gaussian of width $\delta\nu \simeq 1$ and two diracs at positions $\nu = \pm 3$. The product of $\hat{x}(\nu)$ by $\hat{R}(\nu)$ has almost no effect on the Gaussian ($\hat{R}(\nu) \simeq 1$ over its entire width) but multiplies the two diracs by $\hat{R}(3) \simeq 3.10^{-4}$. In direct space, the filtered signal is written [4] $y(t) \simeq a_1 \exp(-t^2) + a_0 \hat{R}(3) \cos(2\pi\nu_0 t)$. We observe that the cosine term has almost disappeared.

In signal processing, low pass filtering is very useful to reduce noise

3.5 Correlations and power spectra

Throughout this paragraph we will assume that the functions with which we are working are square summable, i.e.

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt \quad \text{not infinite} \tag{3.91}$$

In physics we then speak of *finite energy* signals. Indeed the quantity $|f(t)|^2$ often represents a power. For example if f denotes an electric field, then $|f|^2$ is proportional to the electromagnetic power associated with f, and the quantity $\int_{-\infty}^{\infty} |f(t)|^2 \, dt$ represents an energy.

4. The expression given for $y(t)$ is an approximation which assumes that the transfer function is strictly equal to 1 over the entire support of the Gaussian. The exact expression of $y(t)$ is much more complicated.

3.5.1 Correlation functions

Cross-correlation

We call *correlation function* or *cross-correlation* of two signals f and g the following integral :

$$\boxed{C_{fg}(\tau) = \int_{-\infty}^{\infty} \overline{g(t)}\, f(t+\tau)\, dt} \tag{3.92}$$

It is easy to see that C_{fg} can be put in the form of a convolution product

$$C_{fg}(\tau) = [\bar{g} \,*\, f_{-}](-\tau) \tag{3.93}$$

with $f_{-}(t) = f(-t)$. And we notice that when we permute the order of the functions f and g we obtain the following identity

$$C_{gf}(\tau) = \overline{C_{fg}(-\tau)} \tag{3.94}$$

In the case of real and even signals, the cross-correlation of f and g is simply equal to the convolution product of f by g.

Autocorrelation

When f and g are identical, we speak of *autocorrelation function* or simply *autocorrelation* of the signal f. The autocorrelation is defined by the following integral :

$$\boxed{C_{f}(\tau) = \int_{-\infty}^{\infty} \overline{f(t)}\, f(t+\tau)\, dt \;=\; C_{ff}(\tau)} \tag{3.95}$$

and we have

$$C_f(\tau) = \overline{C_f(-\tau)} \tag{3.96}$$

that is, the autocorrelation has an even real part and an odd imaginary part (Hermitian function). The following two properties are of interest :

— If f is real, then C_f is **real pair**

— If f is **real pair** then $C_f = f * f$

Example : Calculation of the autocorrelation of the sum of two Diracs. Consider the function $\upuparrows(t) = \delta(t-t_0) + \delta(t+t_0)$. It is real and even, so $C_{\upuparrows}(\tau) = [\upuparrows * \upuparrows](\tau)$. It comes :

$$C_{\upuparrows}(\tau) = \underbrace{\delta(t-t_0) * \delta(t-t_0)}_{(1)} + \underbrace{\delta(t+t_0) * \delta(t+t_0)}_{(2)} + 2\;\underbrace{\delta(t-t_0) * \delta(t+t_0)}_{(3)} \tag{3.97}$$

To calculate the 3 terms, we can use the property $f(t) * \delta(t+a) = f(t+a)$. Then,

$$\begin{aligned} (1) &= \delta(t-t_0) * \delta(t-t_0) = \delta(t-2t_0) \\ (2) &= \delta(t+t_0) * \delta(t+t_0) = \delta(t+2t_0) \\ (3) &= \delta(t-t_0) * \delta(t+t_0) = \delta(t) \end{aligned} \tag{3.98}$$

We obtain

$$C_{\upuparrows}(\tau) = 2\delta(t) + \delta(t-2t_0) + \delta(t+2t_0) \tag{3.99}$$

i.e. the autocorrelation of a sum of two Diracs is 3 Diracs : one at the origin and the two others at positions $\pm 2t_0$, see figure 3.14.

Application 1 : Measurement of the size of structures by autocorrelation

Consider the example of the signal $f(t)$ in figure 3.15-e. It consists of a series of Gaussian pulses centered at arbitrary positions. Each pulse has a width s. Graphs (a) to (d) of figure 3.15 illustrate how the autocorrelation of $f(t)$ is calculated from its definition (eq. 3.95) . This is the integral of the superposition of $f(t)$ (in red on the graphs) and of $f(t+\tau)$ (in green). When the offset τ increases, the overlap of the green and red signals decreases and the autocorrelation decreases (graph (f)). The first minimum of the autocorrelation is reached when $\tau = s$, allowing the measurement of s. This method is very effective, especially when the number of pulses is large, or when the signal is noisy. In astronomy, it finds applications in speckle interferometry to measure the diameters of stars, for example.

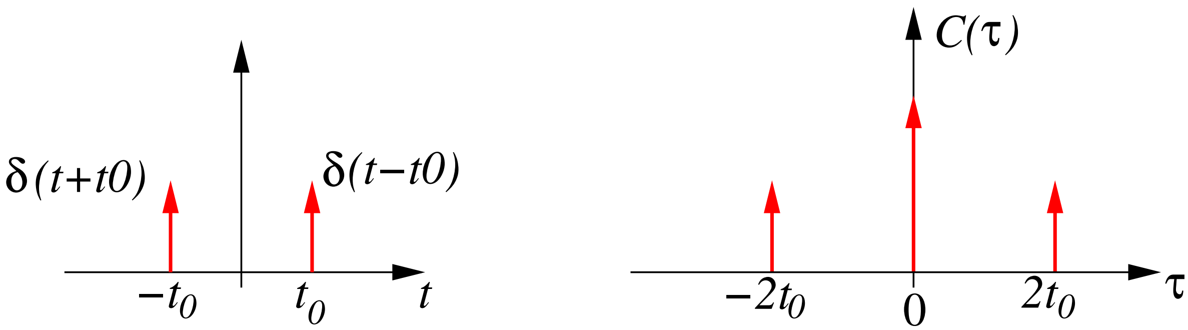


FIGURE 3.14 – On the left, the sum of two Diracs peaks with integral 1, centered at $\pm t_0$. On the right, the autocorrelation is a symmetric function, sum of three Diracs centered at 0 and $\pm 2t_0$. The central peak has an integral twice as large as the side peaks. The distance between the central peak and one of the side peaks is equal to the separation between the Diracs in the function, i.e. $2t_0$.

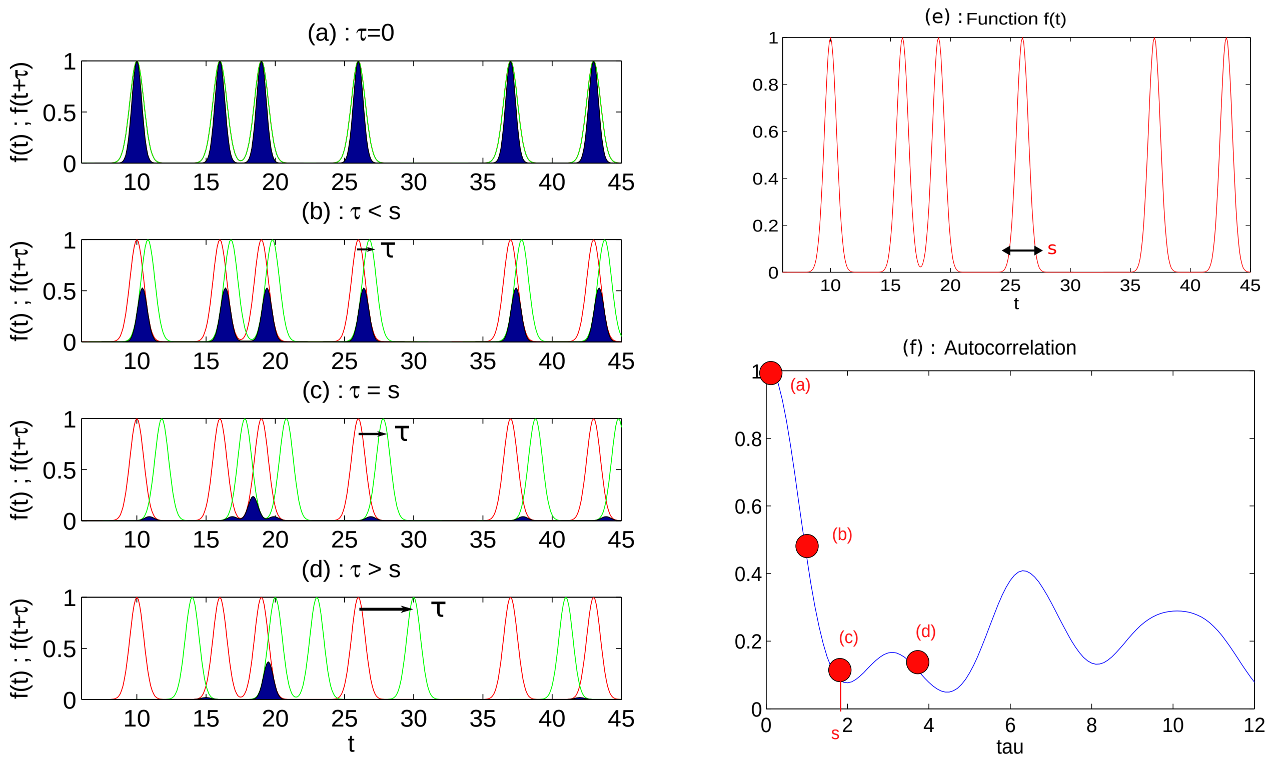


FIGURE 3.15 – Example of calculation of an autocorrelation for a signal $f(t)$ made up of a series of pulses of width s. Graphs (a) to (d) show the superposition of $f(t)$ and $f(t+\tau)$ for 4 values of τ. The graph (f) shows the autocorrelation as a function of τ. The red discs correspond to each of the cases (a) to (d).

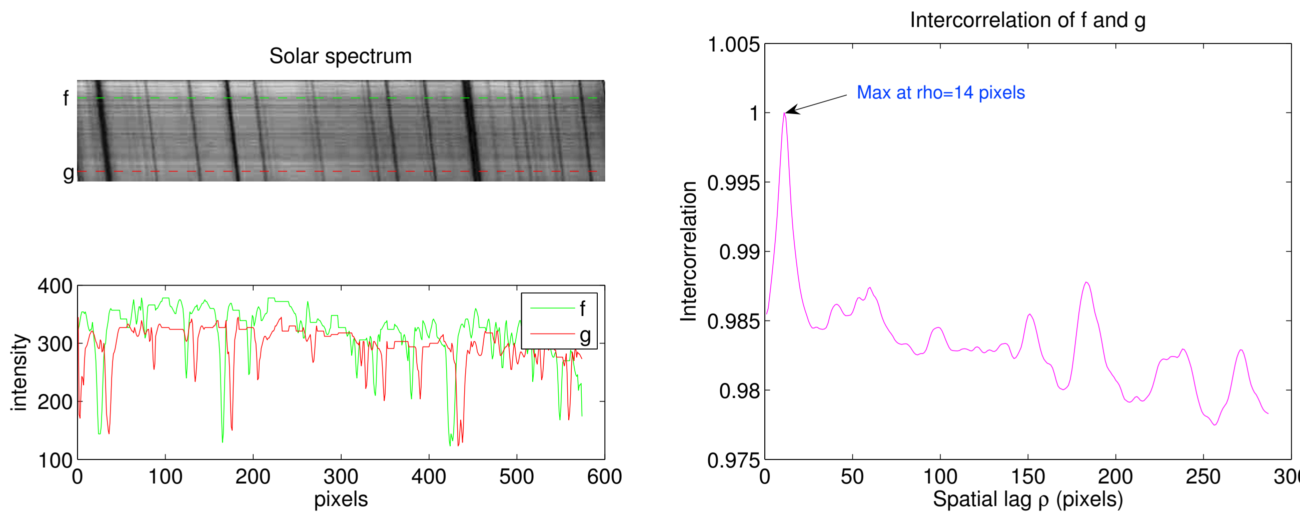


FIGURE 3.16 – Top left : a solar spectrum with tilted lines. Bottom left : intensity profile of the spectrum on the two lines f and g in dotted lines : they are shifted by a quantity Δ. On the right, the cross-correlation of f and g.

Application 2 : Measurement of a shift by cross-correlation

Another very useful application of correlation functions is the measurement of a shift between two identical signals, one of which is shifted with respect to the other by a certain amount Δ. For example on the image of the solar spectrum shown in fig. 3.16, the lines of the spectrum appear tilted. The graph at the bottom left represents the intensity profile of two lines denoted f and g and shows the offset Δ. The graph on the right is the cross-correlation $C_{fg}(\rho)$ between the two lines f and g : it shows a pronounced maximum for the value $\rho = 14$ pixels, which is precisely the offset between f and g.
Other famous applications of correlation functions include optical character recognition on printed pages. The principle is to calculate the cross-correlation of the image of the printed page with the image of each letter of the alphabet. Cross-correlation maxima give the position of the letters.

Degree of coherence

Consider two summable square functions f and g. These functions verify the Schwarz inequality :

$$\left| \int_{-\infty}^{\infty} \overline{f(t)}\, g(t)\, dt \right|^2 \leq \int_{-\infty}^{\infty} |f(t)|^2\, dt \,.\, \int_{-\infty}^{\infty} |g(t)|^2\, dt \tag{3.100}$$

This inequality is the equivalent for functions of the vector relation

$$|\vec{a}.\vec{b}| \leq \| \vec{a} \| \,.\, \| \vec{b} \| \tag{3.101}$$

It allows to write

$$\left| \int_{-\infty}^{\infty} \overline{g(t)}\, f(t+\tau)\, dt \right|^2 \leq \int_{-\infty}^{\infty} |f(t)|^2\, dt \,.\, \int_{-\infty}^{\infty} |g(t)|^2\, dt \tag{3.102}$$

i.e.

$$|C_{fg}(\tau)| \leq \sqrt{C_f(0).C_g(0)} \tag{3.103}$$

We call **degree of coherence** of the functions f and g the quantity

$$\boxed{\gamma_{fg}(\tau) = \frac{C_{fg}(\tau)}{\sqrt{C_f(0).C_g(0)}}} \tag{3.104}$$

with $0 \leq |\gamma_{fg}(\tau)| \leq 1$. This degree of coherence measures the similarity between two functions. It is 1 for $\tau = 0$ when $f = g$, and less than 1 otherwise.

Note : this is the same idea as the correlation coefficient of two centered random variables X and Y. Their covariance is written $\langle X.Y\rangle$ and is the analogue of the cross-correlation in the case of functions. The Schwarz inequality is written $\langle X.Y\rangle \leq \sigma_X\sigma_Y$ with $\sigma_X^2 = \langle X^2\rangle$ the variance of X. The correlation coefficient r of X and Y is

$$r = \frac{\langle X.Y\rangle}{\sigma_X\sigma_Y}$$

with $r \leq 1$. It is the analog of the degree of coherence in the case of functions.

3.5.2 Power spectra

We call **cross-spectrum** of two functions f and g the quantity

$$\boxed{W_{fg}(\nu) = \hat{f}(\nu).\overline{\hat{g}(\nu)}} \tag{3.105}$$

and in the case where $f = g$, we call **power spectrum** of f the quantity

$$\boxed{W_f(\nu) = |\hat{f}(\nu)|^2} \tag{3.106}$$

It represents the *power spectral density*, so that the elementary power contained in a spectral interval of width $d\nu$ around frequency ν is $dW = W_f(\nu)d\nu$.

3.5.3 Wiener-Khinchin and Parseval theorems

Wiener-Khinchin theorem

Let's consider the cross-correlation of two two functions f and g :

$$C_{fg}(\tau) \;=\; [\bar{g} \,*\, f_-](-\tau) \tag{3.107}$$

we calculate its Fourier transform :

$$\mathcal{F}[C_{fg}(\tau)] = [\overline{\hat{g}_-}.\hat{f}_-](-\nu) \;=\; \hat{f}(\nu).\overline{\hat{g}(\nu)} \tag{3.108}$$

after applying equations 3.18 and 3.19. Hence the important property, known as the Wiener-Khinchin theorem, that **the cross-spectrum of two functions is the FT of their cross-correlation** :

$$\boxed{W_{fg}(\nu) = \mathcal{F}[C_{fg}(\tau)]} \tag{3.109}$$

Similarly, when $f = g$, **the power spectrum is the FT of the autocorrelation** :

$$\boxed{W_f(\nu) = \mathcal{F}[C_f(\tau)]} \tag{3.110}$$

Parseval's theorem

It is also known as theorem of Parseval-Plancherel. Plancherel having generalized to all summable square functions the result obtained by Parseval in the case of periodic functions (assuming a Fourier series expansion). To derive it, we can invert equation 3.109

$$C_{fg}(\tau) = \mathcal{F}^{-1}[W_{fg}(\nu)] \;=\; \int_{-\infty}^{\infty} \hat{f}(\nu)\,\overline{\hat{g}(\nu)}\,e^{2i\pi\nu\tau}d\nu \tag{3.111}$$

expanding the quantity $C_{fg}(\tau)$ we have

$$\int_{-\infty}^{\infty} \overline{g(t)}\,f(t+\tau)\,dt \;=\; \int_{-\infty}^{\infty} \hat{f}(\nu)\,\overline{\hat{g}(\nu)}\,e^{2i\pi\nu\tau}d\nu \tag{3.112}$$

For $\tau = 0$ we then obtain the Parseval-Plancherel identity

$$\boxed{\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt \;=\; \int_{-\infty}^{\infty} \hat{f}(\nu)\,\overline{\hat{g}(\nu)}\,d\nu} \tag{3.113}$$

and in the case where $f = g$:

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2\,dt \;=\; \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2\,d\nu} \tag{3.114}$$

This equality offers two expressions to write the total energy contained in the signal f. One as a time integral of the power $|f(t)|^2$, the other as a frequency integral. The two are equal and Parseval's theorem is a way of expressing the conservation of energy when going from the direct plane to the Fourier plane.

Example of application : calculation of integrals Some integrals are easier to compute using Parseval's identity. For example, consider the function $f(t) = \mathrm{sinc}(\pi t)$. Its FT is $\hat{f}(\nu) = \Pi(\nu)$. We can then calculate the following integral, applying the equation3.114

$$\int_{-\infty}^{\infty} \mathrm{sinc}(\pi t)^2 \, dt \;=\; \int_{-\infty}^{\infty} \Pi(\nu)^2 \, d\nu \;=\; 1 \tag{3.115}$$

while the direct calculation of $\int_{-\infty}^{\infty} \mathrm{sinc}(\pi t)^2 \, dt$ is much more difficult.

3.5.4 Uncertainty relations

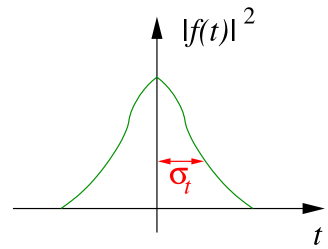
We saw (eq. 3.22) that $f\left(\frac{t}{a}\right) \stackrel{\mathcal{F}}{\longrightarrow} |a| \, \hat{f}(a\nu)$. This is the dilation-compression property (a wide function in direct space is narrow in Fourier space). Hence the naive idea that the product of the widths in the two spaces must be preserved.
This idea has applications in quantum physics. Thus, the localization of a particle (variable x) is defined by the width of its probability density of presence $|\psi(x)|^2$ with $\psi(x)$ the wave function . In Fourier space (using the variable u=spatial frequency, or the variable $k = 2\pi u$=wave number) the quantity $|\hat{\psi}(u)|^2$ has a width σ_u which is inversely proportional to σ_x, so that

$$\sigma_u . \sigma_x = Cte$$

The momentum p of a particle is defined from the wave number k by the de Broglie relation $p = \hbar k = hu$. We then have $\sigma_p . \sigma_x = Cte$, i.e. a well localized particle (σ_x weak) has an uncertain momentum (σ_p large). This is Heisenberg's uncertainty principle.

Similarly, we define an *uncertainty relation* associated with the widths of the same function in direct space and in Fourier space. Consider a function $f(t)$, one way to define its width σ_t is to calculate the following second moment (variance) :

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} t^2 \, |f(t)|^2 \, dt}{\int_{-\infty}^{\infty} |f(t)|^2 \, dt} \tag{3.116}$$



and similarly in Fourier space :

$$\sigma_\nu^2 = \frac{\int_{-\infty}^{\infty} \nu^2 \, |\hat{f}(\nu)|^2 \, dt}{\int_{-\infty}^{\infty} |\hat{f}(\nu)|^2 \, d\nu} \tag{3.117}$$

The application of the following three properties :
- Parseval's identity,
- FT of $t.f(t)$ (eq. 3.43)
- Schwarz inequality

allows to show (cf Roddier, "Distributions and transformation of Fourier") the following uncertainty relation :

$$\sigma_t . \sigma_\nu \geq \frac{1}{4\pi} \tag{3.118}$$

Chapter 4

Fourier series — Sampling

4.1 Fourier series

In this paragraph, we will deal with periodic functions and their Fourier transforms. We will show that the frequency spectra of periodic functions are composed of peaks (harmonic components). We will also show that any periodic function can be put in the form of a discrete sum of sinusoids.

4.1.1 FT of the Dirac comb

Consider the Dirac comb Ш(t) of period 1. Its FT is calculated as follows :

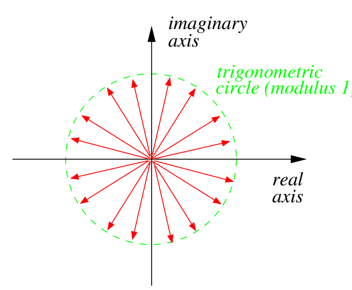
$$\begin{aligned} F(\nu) &= \int_{-\infty}^{\infty} Ш(t)\, e^{-2i\pi\nu t}\, dt \qquad (4.1) \\ &= \int_{-\infty}^{\infty} e^{-2i\pi\nu t} \sum_{n=-\infty}^{\infty} \delta(t-n)\, dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2i\pi\nu t}\, \delta(t-n)\, dt \\ &= \sum_{n=-\infty}^{\infty} e^{-2i\pi\nu n} \end{aligned}$$

$F(\nu)$ appears as a series. It has the following properties :

— F is a periodic function of period 1 (changing $\nu \to \nu+1$ does not change $F(\nu)$). This makes it possible to reduce its study to the interval $[-\frac{1}{2}, \frac{1}{2}[$.

— When $\nu \neq 0$, we notice that $F(\nu)$ is the sum of an infinity of complex numbers of modulus 1 (the sketch on the right shows the addition of the Fresnel vectors in the plane complex). These numbers cancel out two by two and the sum is zero. We thus have $F(\nu) = 0 \quad \forall \nu \neq 0$.

— Lorsque $\nu = 0$, la série diverge et tend vers l'infini.



This gives it the appearance of a Dirac comb of period 1. We will show that this is the case, by establishing an approximate expression of $F(\nu)$ near zero. We first introduce the function

$$F_N(\nu) = \sum_{n=-N}^{N} e^{-2i\pi\nu n} \qquad (4.2)$$

with N integer, which tends to $F(\nu)$ when $N \to \infty$. This function can be written in a more compact form using the formula for the sum of a geometric series of common ratio q :

$$\sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q} \qquad (4.3)$$

It comes :

$$F_N(\nu) = \frac{1-e^{-2i\pi\nu(N+1)}}{1-e^{-2i\pi\nu}} \;+\; \frac{1-e^{2i\pi\nu(N+1)}}{1-e^{2i\pi\nu}} \;-\; 1 \tag{4.4}$$

which takes the for, using the relation $1 - e^{-2ix} = 2i\sin(x)\,e^{-ix}$:

$$F_N(\nu) = e^{-i\pi\nu N}\frac{\sin[(N+1)\pi\nu]}{\sin(\pi\nu)} \;+\; e^{i\pi\nu N}\frac{\sin[(N+1)\pi\nu]}{\sin(\pi\nu)} - 1 \tag{4.5}$$

so :

$$F_N(\nu) = 2\cos(\pi\nu N)\,\frac{\sin[(N+1)\pi\nu]}{\sin(\pi\nu)} \;-\; 1 \tag{4.6}$$

When $N \to \infty$, $N+1 \simeq N$ and $\sin[(N+1)\pi\nu] \simeq \sin(\pi\nu N)$. The trigonometric identity $\sin(2x) = 2\sin x\cos x$ allows to simplify $F_N(\nu)$ in

$$F_N(\nu) \simeq \frac{\sin(2\pi\nu N)}{\sin(\pi\nu)} \;-\; 1 \tag{4.7}$$

The numerator has a period of $\frac{1}{N}$, which becomes small as N becomes large (rapid oscillations). Near the origin, the denominator simplifies : $\sin(\pi\nu) \simeq \pi\nu$. It comes

$$F_N(\nu) \simeq \frac{\sin(2\pi\nu N)}{\pi\nu} \;-\; 1 \;=\; 2N\mathrm{sinc}(2\pi\nu N) - 1 \;\simeq\; 2N\mathrm{sinc}(2\pi\nu N) \tag{4.8}$$

The function F_N behaves like a cardinal sine whose amplitude tends to infinity and width to 0. By setting $\epsilon = \frac{1}{2N}$, we see that F_N has the form $\frac{1}{\epsilon}\,g\left(\frac{\nu}{\epsilon}\right)$ with g the function $\mathrm{sinc}(\pi\nu)$ with integral 1. This form was encountered in paragraph 1.3.3, we showed that its limit when $\epsilon \to 0$ is the distribution $\delta(\nu)$.
In summary, the function $F(\nu)$, Fourier transform of the comb, is periodic of period 1, and equals $\delta(\nu)$ in the interval $[-\frac{1}{2}, \frac{1}{2}[$. It is therefore a comb of period 1, and we can write :

$$\boxed{Ш(t) \;\xrightarrow{\mathcal{F}}\; Ш(\nu)} \tag{4.9}$$

Similarly, the FT of a comb of period T is

$$\underset{T}{Ш}(t) \;\xrightarrow{\mathcal{F}}\; Ш(T\nu) \tag{4.10}$$

4.1.2 Poisson summation formula

Let f be a periodic function of period T as in the example in figure 4.1. It is the repetition of a *pattern* $\phi(t)$ such that

$$\phi(t) = \left[\begin{array}{ll} f(t) & \text{if } t \in [-\frac{T}{2}, \frac{T}{2}[\\ 0 & \text{otherwise} \end{array}\right. \tag{4.11}$$

or, in more compact $\phi(t) = f(t)\,\Pi(\frac{t}{T})$, the rectangle $\Pi(\frac{t}{T})$ ensuring that the pattern vanishes outside the interval $[-\frac{T}{2}, \frac{T}{2}[$. The pattern is also called *main period*. f can therefore be written as a sum of patterns centered on the values of t that are multiples of the period :

$$\boxed{\begin{array}{l} f(t) = \displaystyle\sum_{n=-\infty}^{\infty} \phi(t - nT) \\ f(t) = \phi(t) * \underset{T}{Ш}(t) \end{array}} \tag{4.12}$$

This expression of $f(t)$ constitutes a first representation in the form of a series. There is another, which we will establish below. The starting point for the calculation is to write $f(t)$ as the inverse FT of $\hat{f}(\nu)$:

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\nu)\,e^{2i\pi\nu t}\,d\nu \tag{4.13}$$

and since $f(t) = \phi(t) * Ш_T(t)$ then $\hat{f}(\nu) = \hat{\phi}(\nu).\,Ш(T\nu)$. Expanding the comb, we get

$$\hat{f}(\nu) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \hat{\phi}\left(\frac{n}{T}\right)\,\delta\left(\nu - \frac{n}{T}\right) \tag{4.14}$$

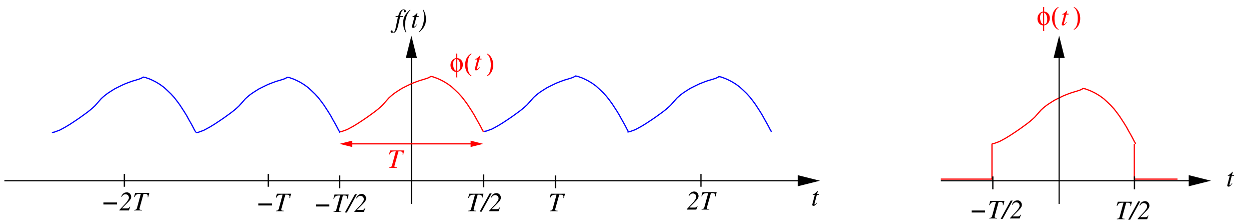


FIGURE 4.1 – On the left a periodic function of period T and its pattern $\phi(t)$ (main period) between $-\frac{T}{2}$ and $\frac{T}{2}$. On the right the graph of the pattern alone. The pattern must vanish outside the range $[-\frac{T}{2}, \frac{T}{2}[$.

and by reverse FT it comes

$$f(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{\phi}\left(\frac{n}{T}\right) e^{2i\pi n \frac{t}{T}} \tag{4.15}$$

which is a second representation of $f(t)$ as a series. This time it is a sum of trigonometric functions, this expansion constitutes a *Fourier series* and will be detailed in the next paragraph 4.1.3. Identifying the two representations (eq. 4.12 and 4.15) yields the identity known as the **Poisson summation formula**

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} \phi(t - nT) \;=\; \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{\phi}\left(\frac{n}{T}\right) e^{2i\pi n \frac{t}{T}}} \tag{4.16}$$

Application : improving convergence of series

By setting $t = 0$ in the Poisson formula, we obtain the following relations :

$$\sum_{n=-\infty}^{\infty} \phi(nT) \;=\; \frac{1}{T} \sum_{n=-\infty}^{\infty} \hat{\phi}\left(\frac{n}{T}\right) \tag{4.17}$$

and in the case where $T = 1$

$$\sum_{n=-\infty}^{\infty} \phi(n) \;=\; \sum_{n=-\infty}^{\infty} \hat{\phi}(n) \tag{4.18}$$

The relation 4.18 makes it possible to accelerate the rate of convergence of series by using the dilation-compression property of FTs (a large function in direct space is narrow in Fourier space). Consider, for example, to calculate the series

$$S = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \tag{4.19}$$

This series converges to $\frac{\pi}{4}$. We note $s_k = \frac{(-1)^k}{2k+1}$ its general term : it is an oscillating term which makes the convergence slow (see fig. 4.2a) : 250 terms are required to obtain a precision of 10^{-3} on the sum S.
To accelerate this convergence by using the relation 4.18, it is necessary to look for a function $s(t)$ which is identified with s_k when $t = k$ integer. By noting that $(1)^k = \sin[(2k+1)\frac{\pi}{2}]$, we can put the series S in the form

$$\begin{aligned} S \;&=\; \frac{\pi}{2} \sum_{n=0}^{\infty} \operatorname{sinc}\left(\frac{n\pi}{2}\right) \;=\; \frac{\pi}{4} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \operatorname{sinc}\left(\frac{n\pi}{2}\right) \\ &=\; \frac{\pi}{4} \left[\underbrace{\sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{n\pi}{2}\right)}_{X} - 1 \right] \end{aligned} \tag{4.20}$$

The term X is of the form $\sum_{n=-\infty}^{\infty} x_n$ with x_n the values of the function $x(t) = \operatorname{sinc}\left(\frac{\pi t}{2}\right)$ for $t = n$ integer. This function extends from $-\infty$ to ∞ and slowly goes to 0 in the distance. Figure 4.2b shows the graph of $x(t)$ and the first terms x_n for $n \geq 0$.

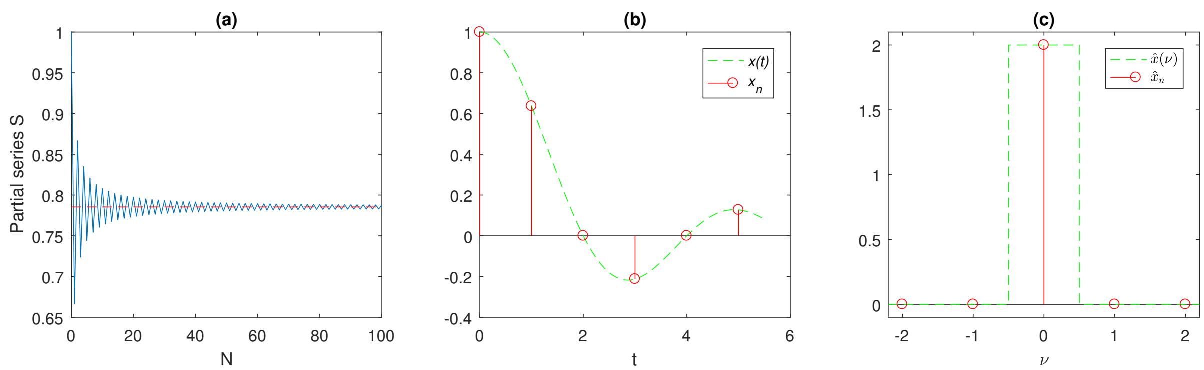


FIGURE 4.2 – Illustration of the convergence acceleration property of the series defined by the equation 4.19. (a) : partial sum of the series S as a function of the number N of summed terms (the dotted line represents the limit of the series). (b) : the term x_n of the series X (eq. 4.20) and the function $x(t)$ which passes through the values x_n for integer t. (c) : the function $\hat{x}(\nu)$ and the terms $\hat{x}_n$ of the series in Fourier space(eq. 4.21).

The TF of $x(t)$ is $\hat{x}(\nu) = 2\Pi(2\nu)$. By applying the equation 4.18 we have

$$X = \sum_{n=-\infty}^{\infty} \hat{x}(n) \ = \ 2 \sum_{n=-\infty}^{\infty} \Pi(2n) \tag{4.21}$$

The only nonzero element of the sequence $\Pi(2n)$ is the term $n = 0$, because the rectangle vanishes when its argument is greater than 1/2 in absolute value. So it comes $X = 2$ and $S = \frac{\pi}{4}$. A single term allowed to sum the entire X series, benefiting from the narrow support of the function $\hat{x}(\nu)$ (see fig. 4.2c). We have thus transformed a series of oscillating principal term having a very slow convergence into a series converging extremely quickly.

4.1.3 Fourier series

The Poisson formula (eq. 4.16) shows that a periodic function of period T takes the form

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{2i\pi n \frac{t}{T}}} \tag{4.22}$$

This expression constitutes the *Fourier series expansion* of the function f. With the coefficient $c_n = \frac{1}{T}\hat{\phi}\left(\frac{n}{T}\right)$. As $\phi = f$ are equal on the interval $[-\frac{T}{2}, \frac{T}{2}[$ and 0 elsewhere, the coefficient c_n is calculated from f :

$$\boxed{c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, e^{-2i\pi n \frac{t}{T}} \, dt} \tag{4.23}$$

The equation 4.22 shows that a periodic function of period T is put in the form of a discrete sum of trigonometric functions of frequencies $0, \pm\frac{1}{T}, \pm\frac{2}{T}, \ldots$called (harmonic components). The frequency $\frac{1}{T}$ is called *fundamental frequency*, it is associated with the *fundamental harmonic.*

The Fourier series is the discrete analogue of the inverse FT integral discussed in paragraph 3.2.8. Retain that :

- A periodic function is written as a Fourier series, it is a *discrete* sum of complex exponentials
- Any function (non periodic) is written as a Fourier integral, it is a *continuous* sum of complex exponentials

Case of real functions

Consider a function f, periodic of period T, and real-valued. It may be convenient to have Fourier series expressions involving only real terms (sines or cosines).

If f real and even : In this case we have $c_n = c_{-n}$ and $f(t)$ is a sum of cosines (which are also real and even functions). By grouping the terms $n > 0$ and $n < 0$ in the equation 4.22, we can write

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(2\pi n \frac{t}{T}\right) \tag{4.24}$$

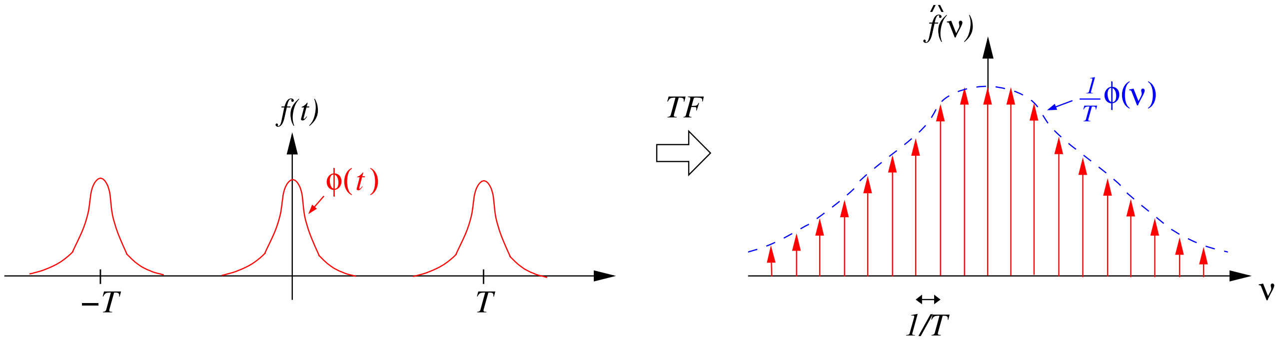


FIGURE 4.3 – On the left, a periodic function with period T and pattern $\phi(t)$. On the right its FT is a comb of period $\frac{1}{T}$ multiplied by the FT of the pattern $\frac{1}{T}\hat{\phi}(\nu)$. Each Dirac has an integral $c_n = \frac{1}{T}\hat{\phi}\left(\frac{n}{T}\right)$ (we sampled the function $\frac{1}{T}\hat{\phi}(\nu)$ with a period $\frac{1}{T}$).

with $a_0 = c_0$ and, for $n \neq 0$:

$$a_n = 2c_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\cos\left(2\pi n\frac{t}{T}\right)\,dt$$

If f real and odd : In this case we have $c_n = -c_{-n}$ and $f(t)$ is a sum of sines (the sine is odd). We obtain

$$f(t) = \sum_{n=1}^{\infty} b_n\,\sin\left(2\pi n\frac{t}{T}\right) \tag{4.25}$$

with $b_n = 2ic_n$ (note that b_n is real and that c_n is pure imaginary).

If any real f : It is always broken down into an even part and an odd part (see paragraph 3.1.2). The Fourier series is written as a sum of sines and cosines :

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n\,\cos\left(2\pi n\frac{t}{T}\right)\;+\;b_n\,\sin\left(2\pi n\frac{t}{T}\right) \tag{4.26}$$

It is in this form that Fourier series were introduced by Fourier in 1822, in his treatise on the Analytical Theory of Heat (eds Firmin Didot, Paris). Today we prefer the more modern approach using complex exponentials, which only require one family of functions and not two to perform the decomposition.

FT of a periodic function

We have already written the FT of a periodic function (eq. 4.14) and have shown that it is a sum of Dirac peaks. The coefficients that weight this sum are precisely the coefficients c_n of the Fourier series expansion (eq. 4.23) :

$$\hat{f}(\nu) = \left[\frac{1}{T}\hat{\phi}(\nu)\right]\,.\,\underset{1/T}{\text{III}}(\nu) \;=\; \sum_{n=-\infty}^{\infty} c_n\,\delta\left(\nu - \frac{n}{T}\right) \tag{4.27}$$

The FT of f is therefore (up to the multiplicative constant $\frac{1}{T}$) the TF of its pattern ϕ multiplied by a comb of period $\frac{1}{T}$. The integral of each Dirac peak is $c_n = \frac{1}{T}\hat{\phi}\left(\frac{n}{T}\right)$, the Fourier coefficient. See figure 4.3 for an illustration. In the sense of distributions, this operation is called **sampling** : this sampling can be schematized as follows :

$$\text{continuous function : } \frac{1}{T}\hat{\phi}(\nu) \quad\Longrightarrow\quad \left\{\frac{1}{T}\hat{\phi}\left(\frac{n}{T}\right)\right\} \text{ : ensemble of values (samples)}$$

We will retain that :

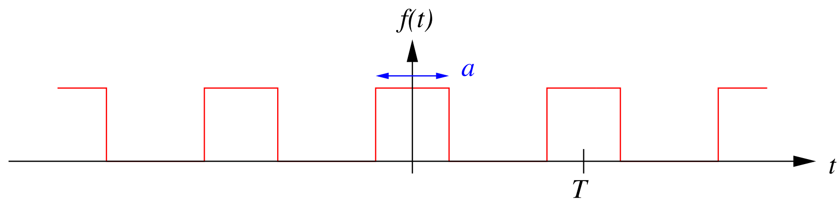
The FT of a periodic function (*period* T) is sampled (*sampling period* $\frac{1}{T}$)

4.1.4 Examples

Crenel function

The pattern here is a rectangular function of width a, the period is T. The function is written

$$f(t) = \sum_{n=-\infty}^{\infty} \prod \left(\frac{t - nT}{a} \right) \qquad \text{motif :} \quad \phi(t) = \prod \left(\frac{t}{a} \right) \tag{4.28}$$



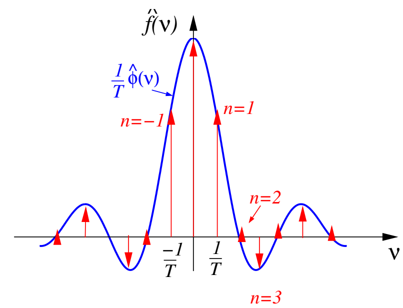
The FT of the pattern is $\hat{\phi}(\nu) = a\,\mathrm{sinc}(\pi\nu a)$, and the coefficient of the Fourier expansion is $c_n = \frac{a}{T}\mathrm{sinc}\left(\pi \frac{na}{T}\right)$. The Fourier series (real expansion on the cosines) is written

$$f(t) = \frac{a}{T} + \sum_{n=1}^{\infty} \frac{2a}{T}\,\mathrm{sinc}\left(\pi\frac{na}{T}\right)\,\cos\left(\frac{2\pi nt}{T}\right) \tag{4.29}$$

It is a sum of cosines of frequencies multiples of $\frac{1}{T}$ (the harmonic components). The constant term $\frac{a}{T}$ is sometimes called "continuous", it represents the average value of the function over its period. It is also a harmonic component of zero frequency. Figure 4.4c shows the shape of the signal reconstructed by the sum of the first 7 terms of the series. Convergence is quite slow in the vicinity of discontinuities where oscillations are observed (Gibbs phenomenon) which will disappear as the number of summed terms increases.

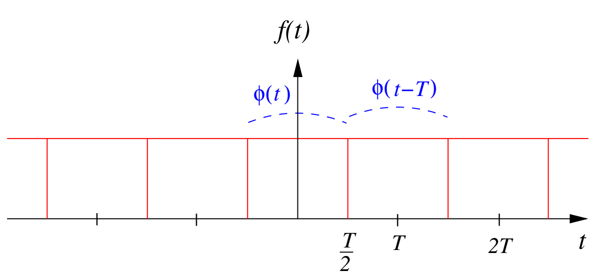
Case $T = 2a$ (Ronchi grating) : we have

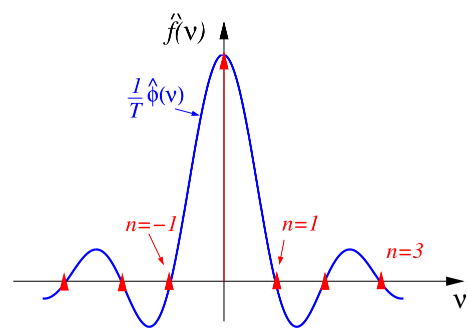
$$\begin{aligned} c_n &= \frac{1}{2}\mathrm{sinc}\left(\pi\frac{n}{2}\right) \\ &= 0 \quad \text{if } n \text{ even (nonzero} \\ &= \frac{(-1)^{(n-1)/2}}{\pi n} \quad \text{if } n \text{ odd} \end{aligned} \tag{4.30}$$



Thus all the even harmonics (except 0) are zero, this is a peculiarity induced by the shape of the pattern $\Pi(\frac{t}{a})$, illustrated by the scheme on the right. The Dirac peaks of the sampling comb corresponding to n even coincide with the zeros of the function $\hat{\phi}(\nu)$ and are absent from the Fourier expansion.

Case $T = a$ **:** it is a constant function of value 1 (the patterns are touching, see on the drawing below on the left). We find that $c_n = \mathrm{sinc}(\pi n)$: all the c_n are null except $n = 0$. the figure below (right) effectively shows that the sampling comb coincides with the zeros of the function $\hat{\phi}(\nu)$, except at the origin. The Fourier series expansion of f is simply $f(t) = 1$.





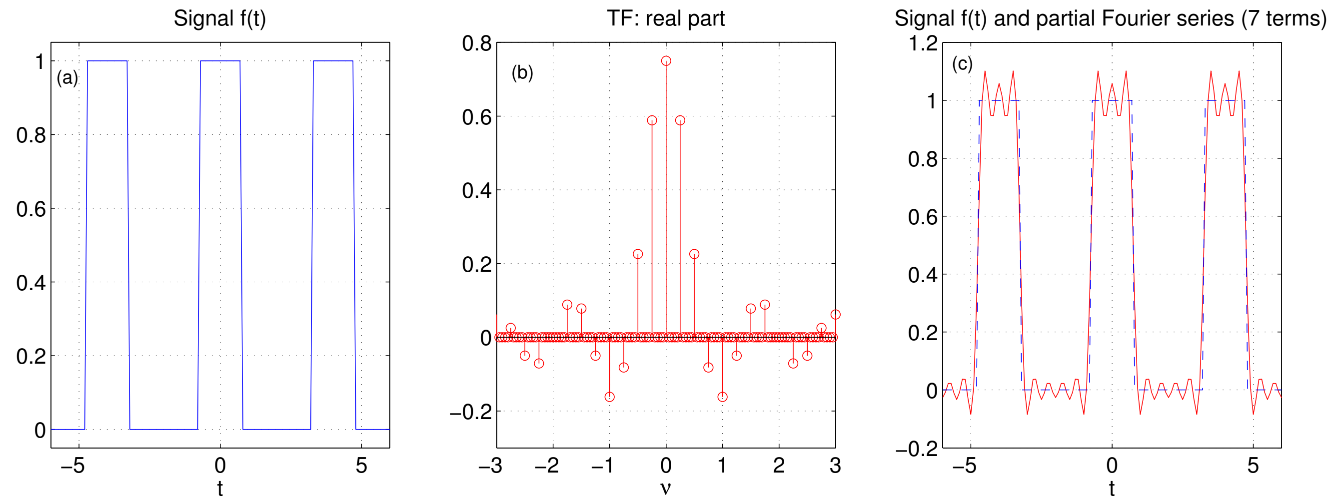
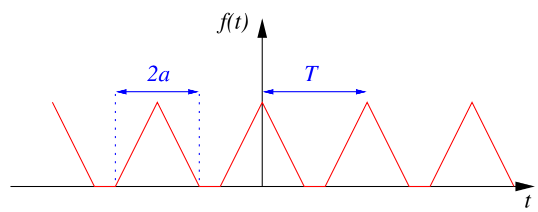


FIGURE 4.4 – (a) : crenel signal of width $a = 1.5$ and period $T = 4$. (b) : its FT (real part), made up of Dirac peaks with integral c_n centered at frequencies $\frac{n}{T}$. (c) : the signal (dotted lines) and the sum of the first 7 terms of its Fourier series(eq. 4.29)
.

Triangular pattern

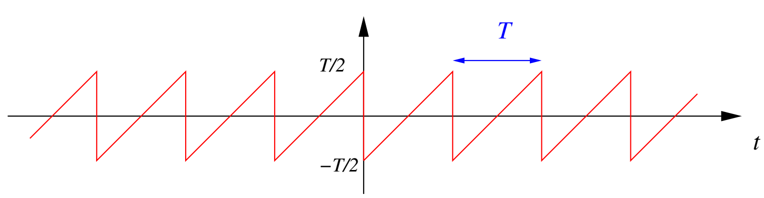


The pattern is $\phi(t) = \Lambda\left(\frac{t}{a}\right)$ with a half the width of the triangle. Its FT is $\hat{\phi}(\nu) = a\mathrm{sinc}^2(\pi\nu a)$ and the Fourier expansion is

$$f(t) = \frac{a}{T} + \sum_{n=1}^{\infty} \frac{2a}{T}\, \mathrm{sinc}^2\left(\pi\frac{na}{T}\right)\, \cos\left(\frac{2\pi nt}{T}\right) \tag{4.31}$$

Here again we obtain a sum of cosines of frequencies multiple of $\frac{1}{T}$. The difference with the development of the crenel function (eq. 4.29) is in the weight of the harmonics (value of c_n). Figure 4.5c shows the shape of the signal reconstructed by the sum of the first 5 terms of the series. Convergence is faster than in the case of the square slot (fig. 4.4c), because the signal does not present any discontinuities.

Sawtooth



The pattern is a line segment with slope 1 on the interval $[0, T[$. It is written

$$\phi(t) = \left(t - \frac{T}{2}\right) \prod \left(\frac{t - \frac{T}{2}}{T}\right) \tag{4.32}$$

To calculate its FT, we can use the relation 3.43, but it is easier to directly calculate the coefficient c_n by the integral of the formula 4.23. We find

$$c_n = i\frac{T}{2\pi n} \tag{4.33}$$

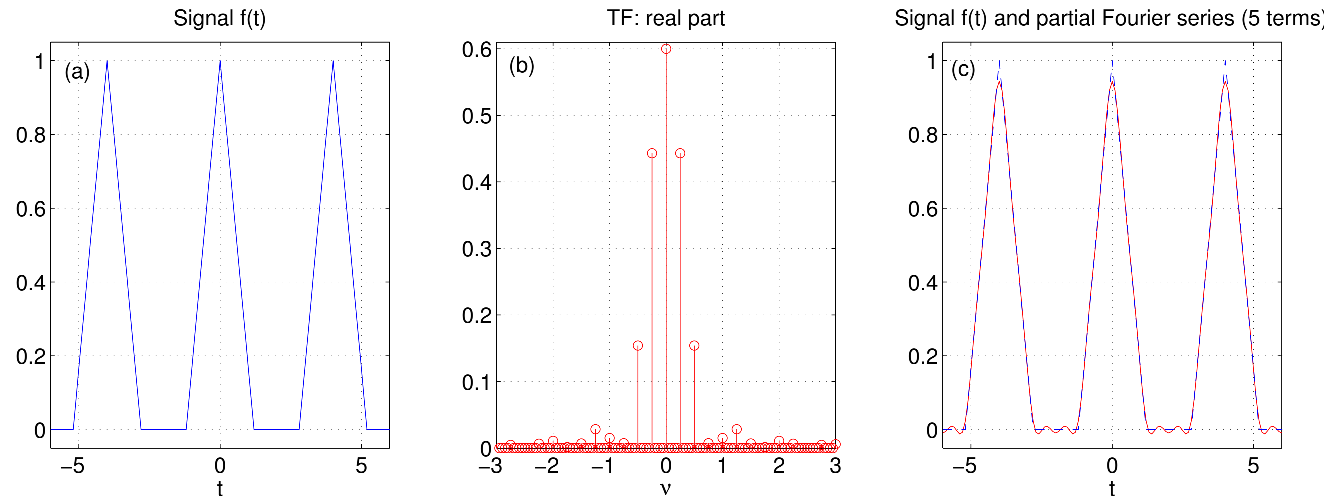


FIGURE 4.5 – (a) : triangular signal of period $T = 4$, the width of the triangles is $2a = 2.4$. (b) : its FT (real part) : the position of the Dirac peaks is the same as in the case of the crenel function (fig. 4.4b), but their integral (coefficient c_n) is different. (c) : the signal (dotted lines) and the sum of the first 5 terms of its Fourier series(eq. 4.31)
.

as expected, the coefficients are pure imaginary (the function f is odd) and verify $c_{-n} = -c_n$ (fig. 4.6b). The Fourier series expansion is written this time as a sum of sines :

$$f(t) = \sum_{n=1}^{\infty} \frac{-T}{\pi n} \sin\left(\frac{2\pi n t}{T}\right) \tag{4.34}$$

Figure 4.6c shows the shape of the signal reconstructed by the sum of the first 5 terms of the series. Convergence is quite fast, except at discontinuities where oscillations are observed as in the case of the crenel function.

4.2 Sampling

4.2.1 Definition

Sampling a function consists of taking a set of values (samples) from the function. This is, for example, what a tape recorder or dictaphone does when recording sound : sound is a continuous function of time, but during recording, values are taken every fraction of a second (generally a value every 1/44000 of a second, we then say that we are "sampling at 44 kHz"). This value of 44 kHz actually corresponds to twice the cutoff frequency of the human ear. Or, equivalently, the 1/44000 second interval is half the ear's "reaction time".
Consider a continuous function $\phi(t)$ from which we take values at $t = t_n = n\,h$ with n an integer and h the "sampling period" (or "sampling interval". Sampling ϕ is to form the sequence $\{\phi(t_n)\}$. But it is sometimes useful to multiply $\phi(t_n)$ by the step h, and to consider instead the sequence $\{f_n\} = \{h.\phi(t_n)\}$. The quantity f_n then corresponds to the area of a rectangle of height $\phi(t_n)$ and width h.
In the sense of distributions, sampling means to consider a function made up of a sum of Dirac peaks centered at t_n and an integral f_n. To the continuous function $\phi(t)$ we associate the distribution

$$f(t) = \sum_{n=-\infty}^{\infty} f_n\,\delta(t - nh) \;=\; \sum_{n=-\infty}^{\infty} \phi(t_n)\,h\,\delta(t - nh) \;=\; \phi(t)\; \text{Ш}\left(\frac{t}{h}\right) \tag{4.35}$$

The comb $\text{Ш}\left(\frac{t}{h}\right)$ has a period h, and each of the peaks has an integral h (note this is not the "usual" comb $\text{Ш}_h(t) = \frac{1}{h}\,\text{Ш}\left(\frac{t}{h}\right)$ which is also periodic with period h but which has peaks of integral 1). And we notice that f and ϕ have the same dimension : this is why we considered the sample sequence $\{h.\phi(t_n)\}$ and not $\{\phi(t_n)\}$. We will retain that :

> Sampling a function $\phi(t)$ with a period h means to multiply it by the comb $\text{Ш}\left(\frac{t}{h}\right)$

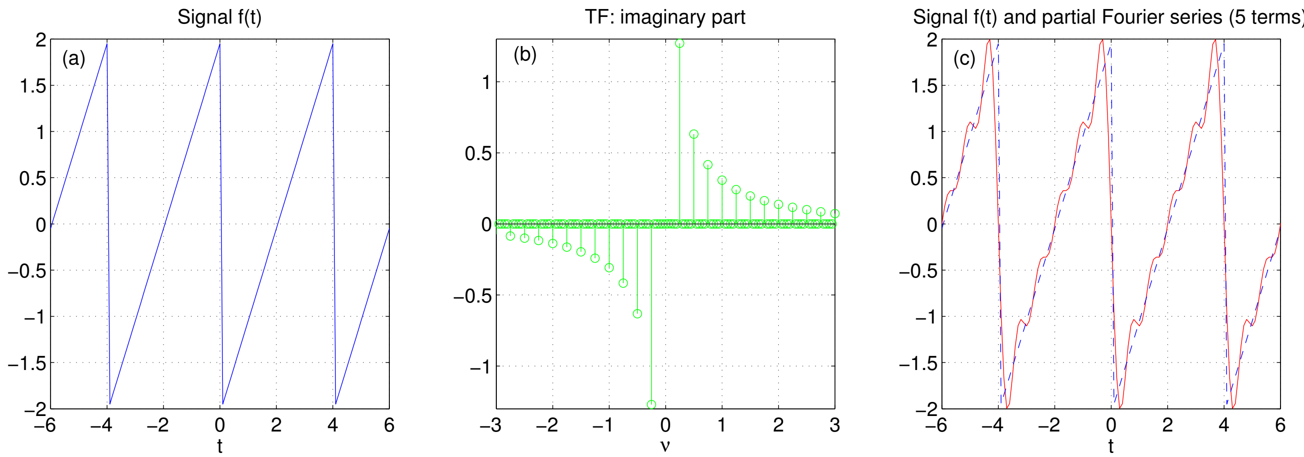


FIGURE 4.6 – (a) : sawtooth signal with period $T = 4$. (b) : its FT (this time we display the imaginary part). (c) : the signal (dotted lines) and the sum of the first 5 terms of its Fourier series (eq. 4.34)
.

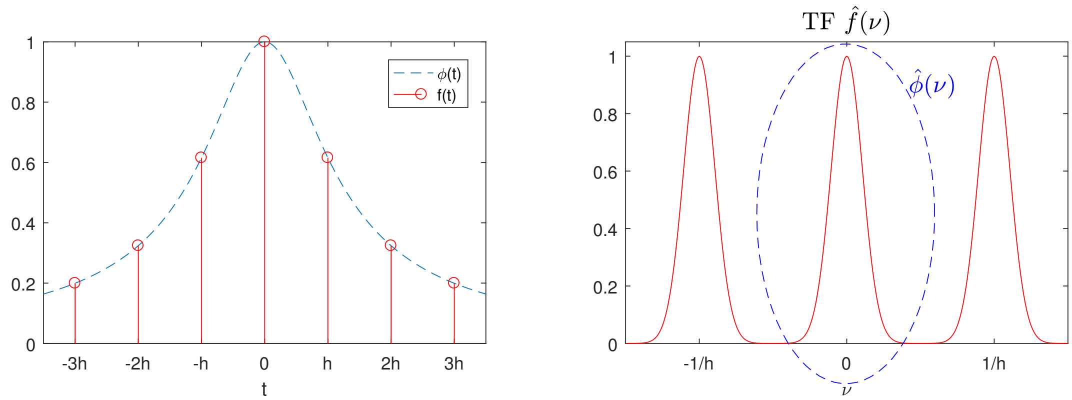


FIGURE 4.7 – Fourier transform of a sampled function. Left : continuous function $\phi(t)$ and its sampled form $f(t)$, made up of a sum of Dirac peaks (the sampling interval is h). Right the FT of $f(t)$. It is a periodic function of period $\frac{1}{h}$ and pattern $\hat{\phi}$.

4.2.2 Fourier transform of a sampled function

We perform a sampling of a function ϕ, and we note $f(t) = \phi(t)\ \text{Ш}\left(\frac{t}{h}\right)$. We want to calculate Fourier transform of f. It comes :

$$\hat{f}(\nu) = \hat{\phi}(\nu) * h\,\text{Ш}(h\nu) \ = \sum_{n=-\infty}^{\infty} \hat{\phi}\left(\nu - \frac{n}{h}\right) \tag{4.36}$$

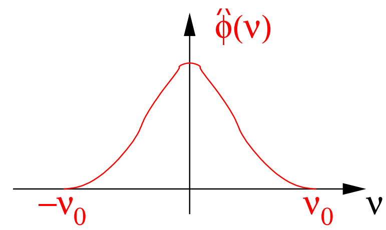
It is a periodization of the function $\hat{\phi}$ with a period $\frac{1}{h}$. We then find the interesting property according to which sampling a function (with a step h) amounts to periodizing its FT (period $\frac{1}{h}$). We will retain that

The FT of a sampled function (with a sampling period h) is periodic (period $\frac{1}{h}$)

It is remarkable to note that this property is the exact symmetry of that encountered in paragraph 4.1.3 : the FT of a periodic function (period T) is sampled (step $\frac{1}{T}$).

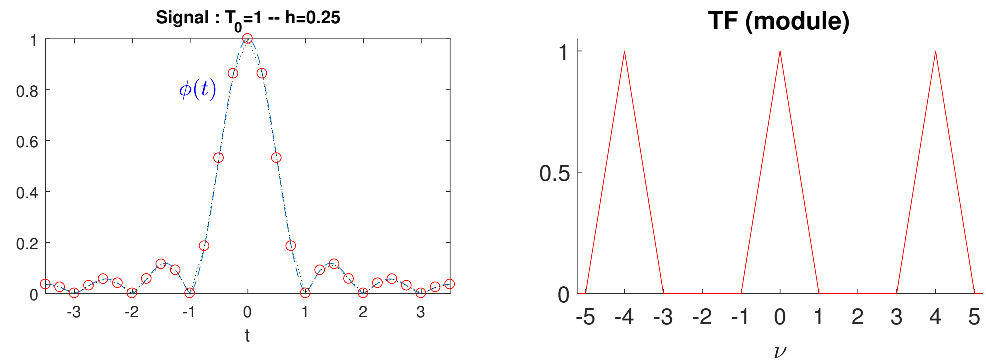
4.2.3 Shannon-Nyquist Theorem

Shannon's theorem, or Shannon-Nyquist, establishes the conditions for optimal sampling of a band-limited signal (ie whose FT has a cut-off frequency). This is the case for the majority of signals encountered in physics. Consider a continuous signal $\phi(t)$ whose FT has bounded support, i.e. it vanishes outside a frequency interval $[-\nu_0, \nu_0]$ (see diagram on the right). The quantity ν_0 is called **cutoff frequency** of ϕ. It corresponds to the frequency of the tightest sinusoid present in the Fourier expansion of ϕ. Its inverse $T_0 = 1/\nu_0$ is called *cutoff period*.



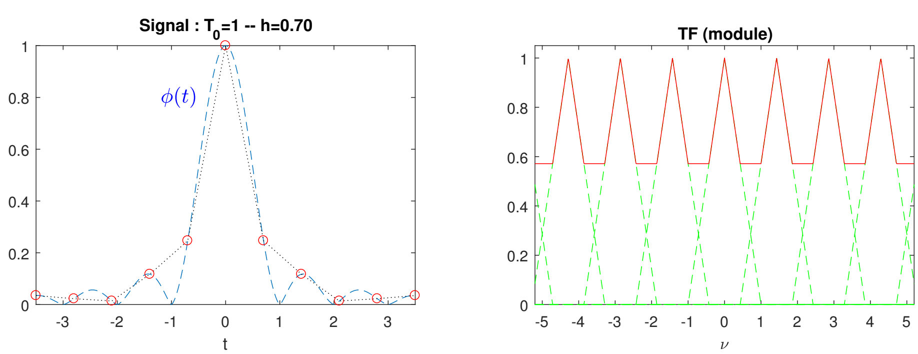
Let h be the sampling step for ϕ. Three cases can be distinguished :

— Case h small ($h \ll T_0$)



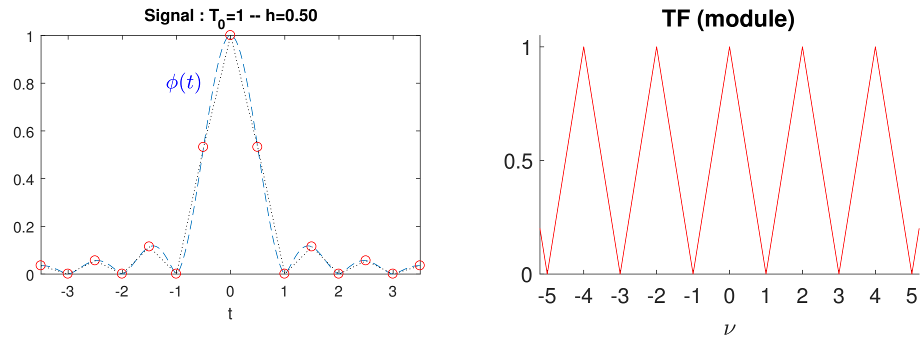
The scheme above shows the effect of sampling on the continuous signal $\phi(t) = \text{sinc}(\pi\nu/T_0)^2$ with $T_0 = 1$. Its FT is the triangle function $\hat{\phi}(\nu) = T_0\Lambda(\nu T_0)$ and has a cutoff frequency $\nu_0 = 1$. On the left : the signal $\phi(t)$ and the sampling points (red circles) for $h = 1/4$. On the right the FT shows the juxtaposition of several patterns $\hat{\phi}$ centered in $0, \pm 1/h$. Note that the different patterns in Fourier space do not overlap. The weaker h, the more distant these patterns are. Between two successive patterns, the FT is zero. This case corresponds to **oversampling**.

— Case h large



The sampled signal is the same as above, but the sampling interval is larger : $h = 0.7$. On the right, the FT of the sampled function (in red) and the different patterns $\hat{\phi}$ which overlap (in green dotted lines). This is the phenomenon of *spectrum aliasing*). We are in the case of a **undersampling**. In direct space (left) the red circles do not correctly sample the fine details of the function ϕ (in particular the local minima of the function sinc^2).

— Optimal case : $h = \frac{T_0}{2}$.



In Fourier space, the patterns touch each other without overlapping. We then speak of **optimal sampling**. This property is known as the Shannon-Nyquist theorem (or the *sampling theorem*). The optimal sampling step, or *Nyquist period* is equal to

$$h_e = \frac{T_0}{2} = \frac{1}{2\nu_0} \tag{4.37}$$

similarly, we define the *Nyquist frequency*, inverse of h_e :

$$\nu_e = 2\nu_0 = \frac{2}{T_0} \tag{4.38}$$

We will retain that :

The Nyquist frequency is equal to twice the cutoff frequency of the signal.

A signal is said to be *correctly sampled* if the sampling frequency is equal to ν_e. A signal is oversampled (resp. undersampled) if the sampling frequency is higher (resp. lower) than ν_e. Thus a sound signal audible by the ear (maximum frequency $\simeq$ 20 kHz) must be sampled at 40 kHz or more (it is generally 44 kHz in classic WAV or MP3 files). Oversampling is not a problem in terms of the storage of the information present in a signal ; on the other hand, it is necessary to avoid undersampling (loss of information, even appearance of parasitic structures as illustrated by the example below.

Example : Sampling a Cosine Function Consider the function $\phi(t) = \cos\left(\frac{2\pi t}{T_0}\right)$. It has a unique period T_0 which is therefore its cutoff period. Its cutoff frequency is $\nu_0 = \frac{1}{T_0}$. Figure 4.8a illustrates oversampling with a large number of points per period, allowing faithful reproduction of signal variations. The case of Shannon sampling corresponds to figure 4.8b. In this case we have two points per period T_0. The graph of the sampled signal looks like a sawtooth, but allows the measurement of T_0. The figure 4.8c is an undersampling with one point per period ($h = T_0$) : the sampled signal is constant, we no longer see the periodicity of ϕ, we can no longer measure T_0. Finally, figure 4.8d shows another case of undersampling with $h = 1.2T_0$. This time the sampled signal shows a period greater than T_0 (a lower frequency which is a noise due to aliasing).
This last case corresponds to the phenomenon of the *airplane propeller*. Observe a rotating airplane propeller : your eye naturally samples the image of this propeller with a rate of about 25 frames per second. If the propeller rotates faster and faster, you will have the impression that it becomes stationary when its rotation period corresponds to the sampling period of the eye (case of the figure 4.8c). If the propeller spins even faster, you'll feel like it's changing direction and spinning slowly. (figure 4.8d).

4.2.4 Unsampling — Shannon Interpolation

Let f be a sampled function with a sampling interval h (sum of Dirac peaks). To build this function, we take the product of a continuous function ϕ by a comb of period h : $f(t) = \phi(t)\ \text{Ш}\left(\frac{t}{h}\right)$. We are interested in "unsampling" the function, i.e. finding ϕ, knowing f. It therefore consists of calculating the values of ϕ between the sampling points, a classic problem of interpolation in numerical analysis. Various methods exist, for example the linear interpolation which use straight lines between the points. We thus obtain an approximation of $\phi(t)$ for all t.
This paragraph presents a method which allows to calculate the *exact* values of the function ϕ between the sampling points. It works under two conditions :

1. The FT $\hat{\phi}(\nu)$ has bounded support with cutoff frequency ν_0
2. The ϕ function is not undersampled : $h \leq \frac{1}{2\nu_0}$

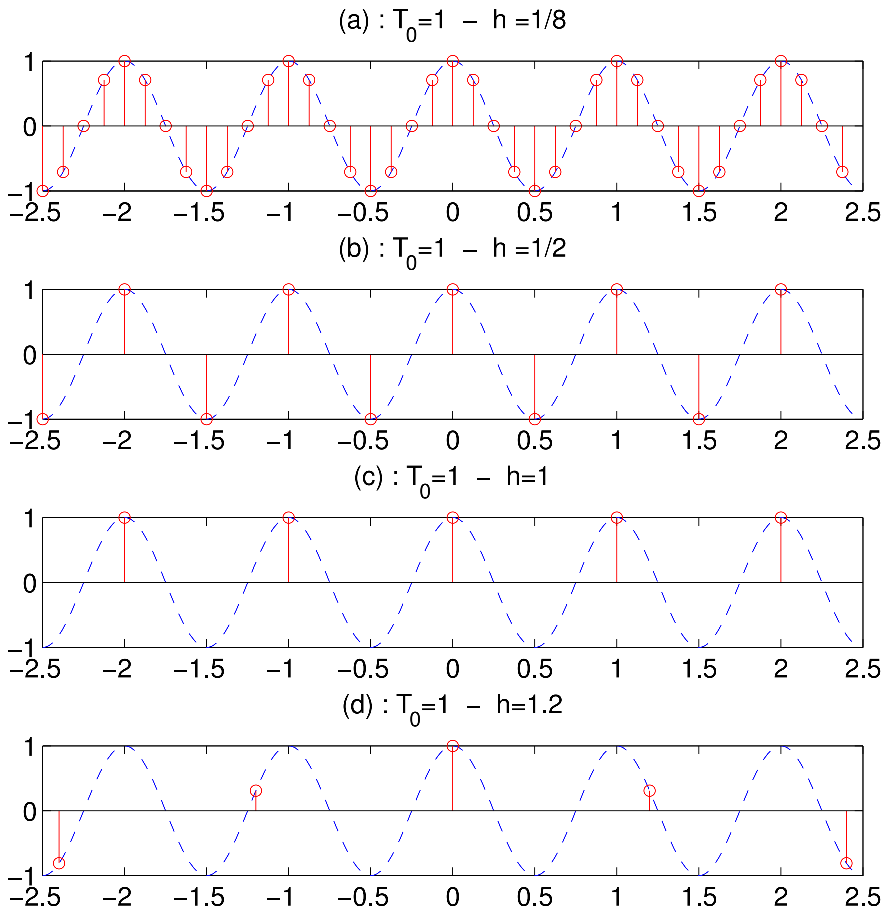
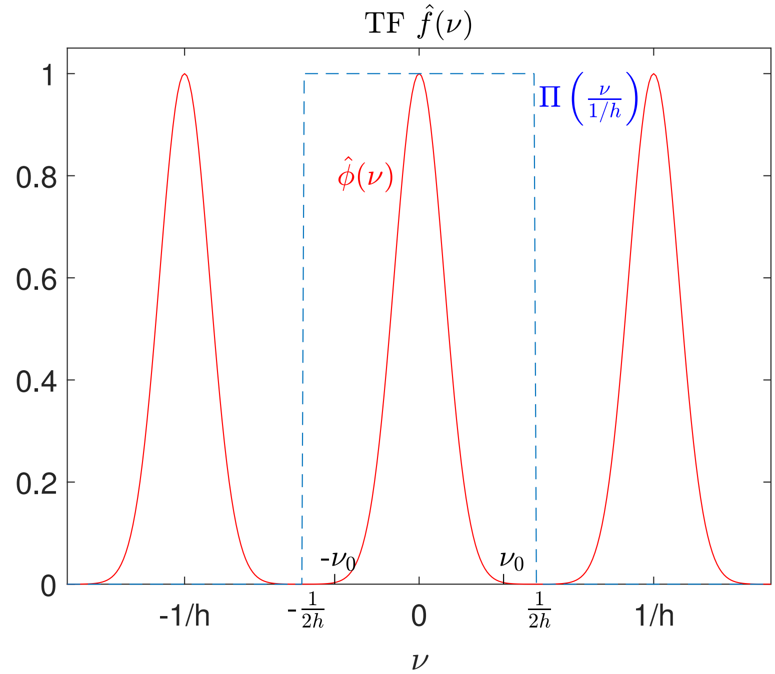


FIGURE 4.8 – Sampling of a cosine function of period $T_0 = 1$. The function is in dotted line, samples are drawn with red circles. (a) : case of oversampling (step $h = 1/8$). (b) : optimal sampling (Shannon condition, not $h = \frac{1}{2}$. (c) undersampling with $h = 1$: the signal seems constant. (d) undersampling with $h = 1.2T_0$: the signal seems to have a much larger period than T_0 (low apparent frequency).

The principle is to make use of the particular form of the FT $\hat{f}(\nu)$, composed of a succession of patterns $\hat{\phi}$ centered in n/h (n integer). If the two conditions above are met, there is no spectrum aliasing and the patterns are disjoint. The idea is then to isolate the central pattern by multiplying $\hat{h}$ by a gate of width $\frac{1}{h}$:



The gate of width $\frac{1}{h}$ is chosen to remove from $\hat{f}$ all side patterns without altering the central pattern. We will therefore have :

$$\hat{\phi}(\nu) = \hat{f}(\nu) \prod \left(\frac{\nu}{1/h}\right) \tag{4.39}$$

This relation then makes it possible to calculate $\phi(t)$ by inverse FT.

Expression of the interpolation in the direct plane

It is a question of obtaining a formula giving $\phi(t)$ for all t by carrying out an inverse FT of equation 4.39. It comes :

$$\begin{aligned} \phi(t) &= f(t) * \frac{1}{h}\,\mathrm{sinc}\left(\pi\frac{t}{h}\right) = \frac{1}{h}\left[\phi(t)\ \mathrm{III}\left(\frac{t}{h}\right)\right] * \mathrm{sinc}\left(\pi\frac{t}{h}\right) \\ &= \sum_{n=-\infty}^{\infty} \phi(nh)\,\delta(t-nh) * \mathrm{sinc}\left(\pi\frac{t}{h}\right) \end{aligned} \tag{4.40}$$

And we get the following formula, known as the *Shannon-Whittaker interpolation formula* :

$$\boxed{\phi(t) = \sum_{n=-\infty}^{\infty} \phi(nh)\ \mathrm{sinc}\left(\frac{\pi}{h}(t-nh)\right)} \tag{4.41}$$

Since $\phi(nh)$ is known (the function is sampled with step h), this relation allows to calculate $\phi(t)$ for all t. Figure 4.9 illustrates this relationship. It shows how, between sample points, the function ϕ can be computed as a sum of sinc functions. For this reason, the sinc is sometimes called *interpolation function.*

Example : doubling the sampling rate of a signal

Consider the following numerical analysis problem : we have a signal with a sampling period h, we want to resample it with a period $h' = \frac{h}{2}$. We can use the Shannon-Whittaker formula (eq. 4.41), but it is more interesting in this case to do this interpolation in the Fourier plane using a fast Fourier transform algorithm. Figure 4.10 illustrates the principle. If the signal $f(t)$ is well sampled (or oversampled), its FT is made up of disjoint patterns, centered at frequencies $\frac{n}{h}$ (n integer), between which the value is 0 (fig. 4.10b). The idea is to move these patterns to frequencies $\frac{n}{h'}$, by inserting zeros between them (*zero-padding* technique, fig. 4.10c). By inverse FT, we then find the signal $f(t)$, sampled with the new interval h'.

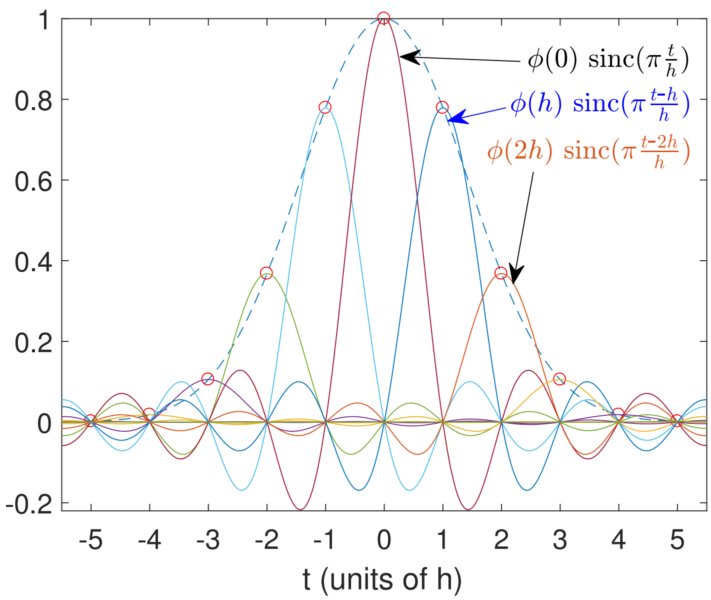


FIGURE 4.9 – Illustration of the Shannon-Whittaker interpolation formula (eq. 4.41). In dotted lines, the function $\phi(t)$. The red circles represent the sample points. The interpolation formula shows that in all t the function ϕ is the sum of an infinity of sinc functions centered on the sampling points, in $t = nh$, weighted by $\phi(nh)$, with n integer from $-\infty$ to ∞. The cases $n = 0$, $n = 1$ and $n = 2$ are highlighted on the graph.

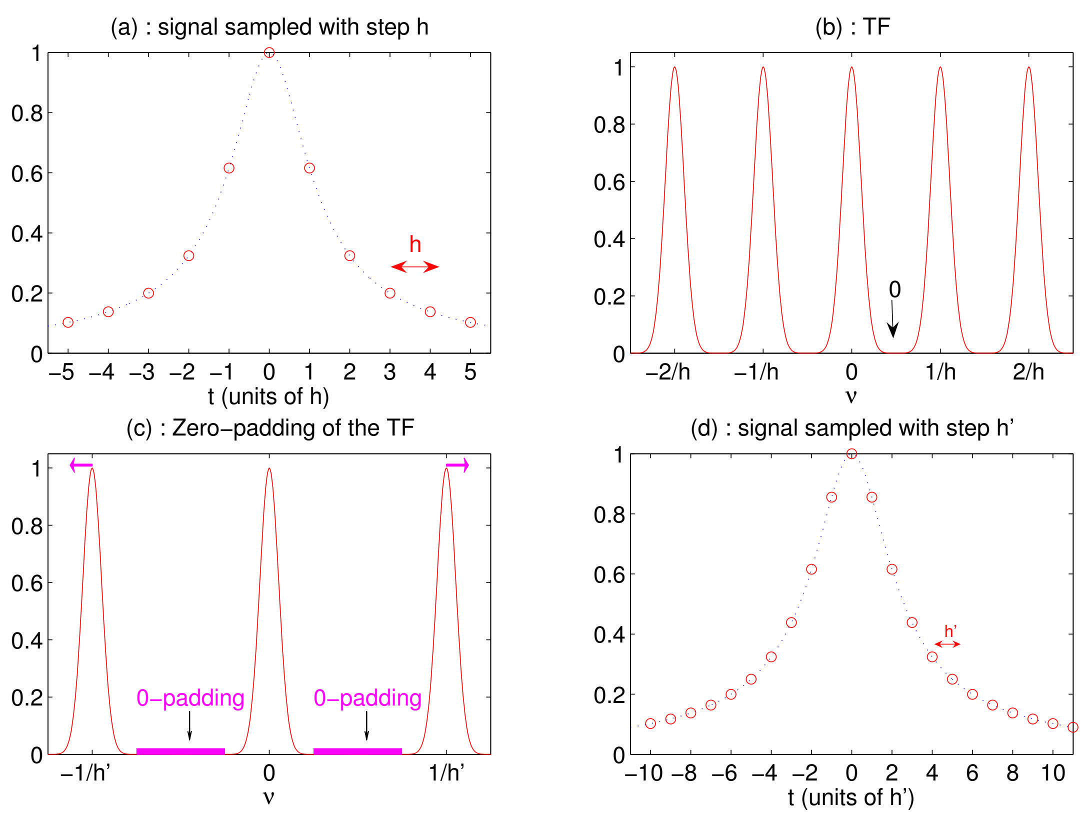


FIGURE 4.10 – Resampling a signal $f(t)$ (of sampling period h) with a period $h' = \frac{h}{2}$. (a) signal $f(t)$. (b) its FT, consisting of disjoint patterns centered at frequencies $\frac{n}{h}$ (n integer). (c) after moving the patterns to frequencies $\frac{n}{h'}$, and zero-padding. (d) after inverse FT, the resampled signal with period h'.

Chapter 5

Exercices

Thanks to my colleagues Jacques-Alexandre Sepulchre, Jean-Pierre Provost and Jean-Louis Meunier with whom I shared these tutorial exercises for many years.

5.1 Dirac δ impulse

5.1.1 Usual functions in signal analysis

The functions defined below are used frequently in the other exercises. Graph these functions when the argument t is replaced by $\dfrac{t-\tau}{T}$.

Heaviside unit step function $H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$

Gate function : $\Pi(t) = \begin{cases} 1 & \text{if } |t| < 1/2 \\ 0 & \text{if } |t| > 1/2 \end{cases}$

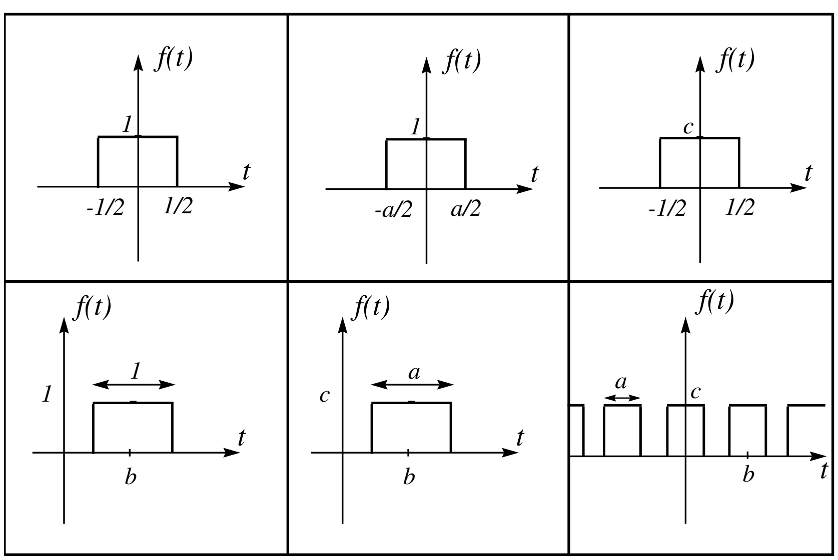
Triangle function : $\Lambda(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$

Cardinal sine : $\text{sinc}(t) = \frac{\sin t}{t}$

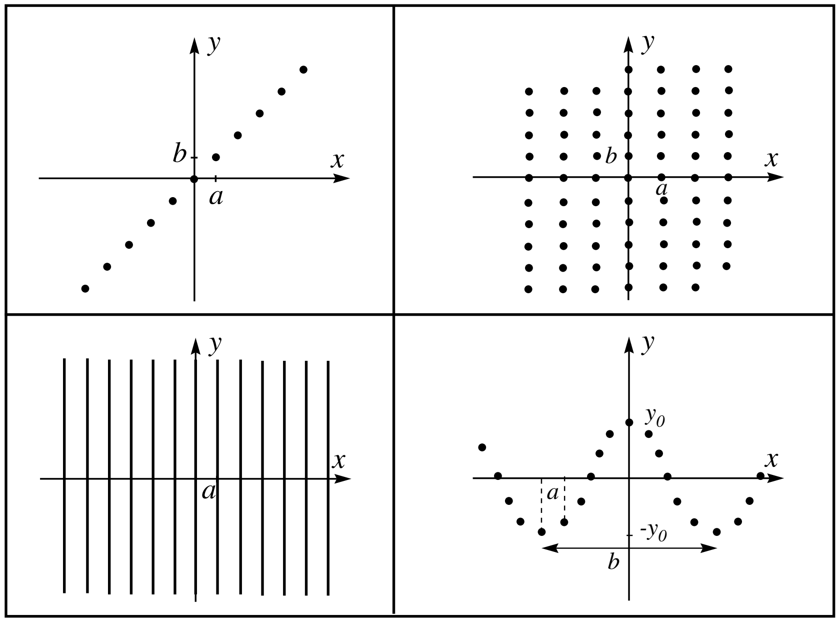
Dirac comb : $\text{Ш}(t) = \sum_n \delta(t-n)$

5.1.2 Translation and dilation of a signal

1. Write the expression of the functions below in terms of the gate function (the last function is periodic) :



2. We define the two-variable Dirac impulse $\delta(x, y) = \delta(x).\delta(y)$. Write the expression for the functions $z = f(x, y)$ below (each black dot represents a delta peak seen from above. For the fourth figure, the function responsible for the oscillations is a cosine.



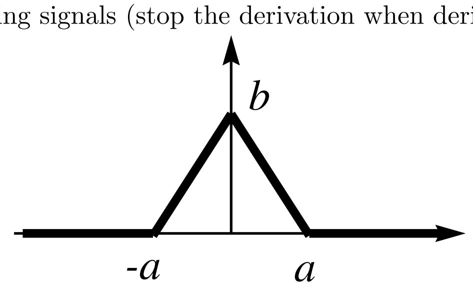
5.1.3 Dirac delta impulse

1. Express Dirac's δ in terms of limits using the following functions :
   — $\frac{1}{1+t^2}$
   — $\text{sinc}(t)$
   — $\text{sinc}^2(t)$
   — $e^{-(t-1)^2}$
2. What is the limit of the sequence of functions $f_n(t) = \frac{n}{\sqrt{\pi}} e^{-n^2 t^2}$?
3. Graphically represent the signal $g(x) = \sum x\delta(x - n)$. Calculate $\int_{-0.5}^{N+0.5} g(x)dx$.
4. With the definition of $\delta(t)$ as the limit of a gate, show that $\int_{t_1}^{t_2} \delta(t)\varphi(t)\, dt = \begin{cases} \varphi(0) & \text{if } t_1 < 0 < t_2 \\ 0 & \text{otherwise} \end{cases}$

5.1.4 Derivatives of discontinuous signals

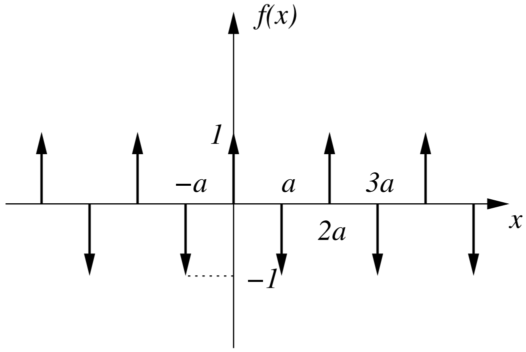
Calculate and plot the derivatives of the following signals (stop the derivation when derivatives of δ appear) :

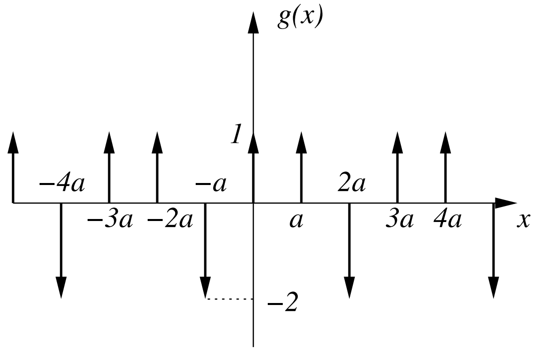
— $\exp -|t|$,
— $H(t)\,\dfrac{\sin \omega t}{\omega} \exp\left(-\dfrac{\gamma t}{2}\right)$,
— $\text{sgn}(t) = \dfrac{t}{|t|}$,
— The triangle function on the right.



5.1.5 Dirac comb

1. Write the expression for the periodic distributions whose graphs are shown below.





2. Plot the function

$$F(x) = \int_0^x Ш(t)\, dt$$

We will study two cases : integration interval $]0, x]$ (limit 0 not included) then $[0, x]$ (limit 0 included).

3. Plot the distribution $f(x) = x\ Ш(x)$ and calculate the integral

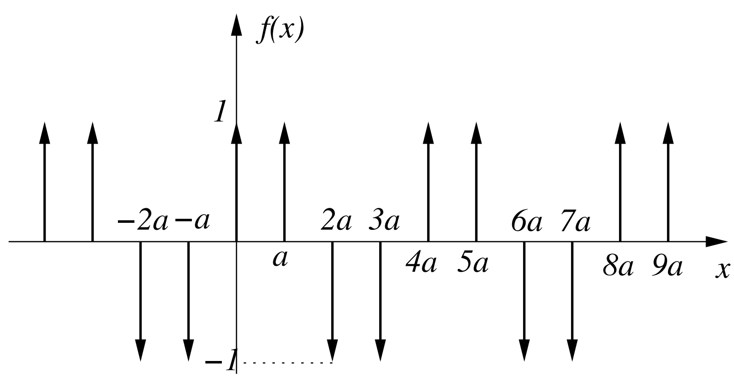
$$I_N = \int_{1/2}^{N+1/2} f(x)\, dx$$

with N positive integer.

4. Let $f(x) = H\left(x + \frac{1}{2}\right)\, e^{-x}\ Ш(x)$. Draw its graph then calculate its integral from $-\infty$ to $+\infty$. We recall that the sum of a geometric series with common ratio $q < 1$ is

$$\sum_{n=0}^{\infty} q^n \;=\; \frac{1}{1-q}$$

5. Consider the periodic distribution below. Give its period and write its mathematical expression using δ distributions and/or combs.



5.1.6 Derivative of δ distribution

1. Write the derivative of the gate function $g(x) = \prod\left(\frac{x}{a}\right)$, with $a > 0$.
2. Let $g_\epsilon(x) = \frac{1}{\epsilon}\prod\left(\frac{x}{\epsilon}\right)$. What is $\lim_{\epsilon\to 0} g_\epsilon(x)$?
3. Write the derivative $g'_\epsilon(x)$ and deduce a possible definition of the derivative δ' of the Dirac distribution. Plot $\delta'(x)$.
4. Let f be a function defined for all real x ; based on the previous question, give the value of the product $f(x)\ \delta'(x)$ for $x \neq 0$. What is the integral $\int_{-\infty}^{\infty} f(x)\ \delta'(x)\ dx$?
5. Calculate $\int_{-\infty}^{\infty} f(y)\ \delta'(x-y)\ dy$. What property of the convolution product do we find ?

5.2 Linear filter and convolution

5.2.1 Convolution product

We define the convolution between two functions f and g by :

$$h(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(t-t')g(t')dt'$$

This operation has the properties of a product.

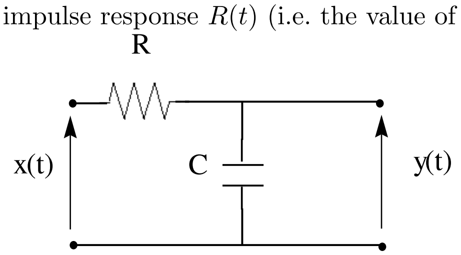
1. Show that :
   — $f * g = g * f$
   — if $f_\tau(t) = f(t+\tau)$, show that $(f*g)_\tau = f_\tau * g = f * g_\tau$
   — $(f*g)' = f' * g = f * g'$ (consider $(f*g)_\tau$ with $\tau \to 0$)
   — $(f*g)(\lambda t) = |\lambda| f(\lambda t) * g(\lambda t)$
2. Calculate and draw the following functions :
   — $\Pi(t) * a\cos(\omega t + \phi)$
   — $\Lambda(t) = (\Pi * \Pi)(t)$
   — $(\Pi * \Pi * \Pi)(t)$
3. Express the following operation,

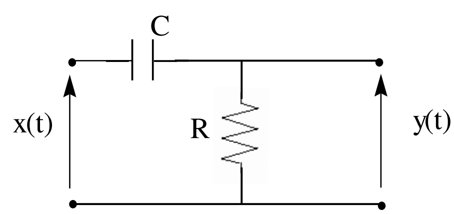
   $$f \to F(t) = \frac{1}{\tau}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} f(t')\,dt'\ ,$$

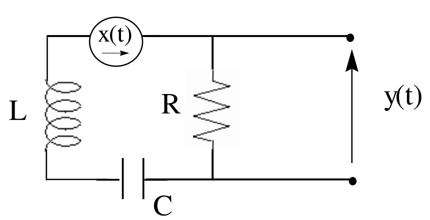
   as a convolution product. What type of signal processing can this operation correspond to ?

5.2.2 Impulse responses

1. We consider the electric circuits below, with $x(t)$ and $y(t)$ the input and output voltages. Using circuit laws, calculate the response $g(t)$ to the Heaviside function (i.e. the value of $y(t)$ when $x(t) = H(t)$), and the impulse response $R(t)$ (i.e. the value of $y(t)$ when $x(t) = \delta(t)$).







On posera : $\gamma = \dfrac{R}{2L}$

$\omega = \sqrt{\dfrac{1}{LC} - \gamma^2}$

2. Same question as above, with RL circuits.
3. A mechanical damper is described by the equation

   $$m(\ddot{y} + \gamma\dot{y} + \omega_0^2 y) \ = \ x(t)$$

   where $x(t)$ is a time-dependent exciting force. γ is the coefficient which expresses the viscosity of the medium and $m\omega_0^2$ the stiffness constant of the spring. We suppose that $\gamma > 2\omega_0$.
   — Show that the system can be interpreted as a linear filter. (explicit the input and output of the filter).
   — Calculate the impulse response of the system.
   — Write the general solution $y(t)$ in the highly damped case. What if the force x is a train of periodic pulses $Ш_T(t)$?

4. Calculate the impulse response $R(x)$ associated with the following differential equation of order 4, where $f(x)$ is the input signal and $y(x)$ is the system output :

$$-\alpha\frac{d^4y}{dx^4}+\beta\frac{d^2y}{dx^2}=f(x),$$

with α and β positive real constants. Make a plot the function $R(x)$.

5.2.3 Photon distribution

The arrival on a sensor of photons coming from a light source can be modeled by a set of discrete pulses similar to Dirac distributions $\delta(t)$. The arrival times of the photons are random, but we will here consider that the photons arrive at perfectly determined instants. We then have the following intensity distribution :

$$f(t)=\sum_n \delta\left(t-n\tau-\epsilon\cos\left(2\pi\frac{n\tau}{T}\right)\right)$$

with $\epsilon<\tau$ and $T\gg\tau$.

1. Plot the function $f(t)$. Is it periodic ?
2. Calculate and plot the function

$$F(t)=\int_0^t f(t')\,dt'$$

3. Let t_{max} and t_{min} be the maximum and minimum time intervals separating the arrival of two successive photons. Calculate t_{max} and t_{min}. N.A. : $\tau=0.1$ s, $\epsilon=0.05$ s, $T=2.05$ s.
4. The photons are observed by a photomultiplier whose impulse response is modeled by a gate of width t_0. The measured light curve is plot on a screen as a function of time. Draw the shape of the light curve if $t_0<t_{min}$.
5. We define the sequence of numbers g_n

$$g_n=\int_{nt_0}^{(n+1)t_0} f(t)\,dt$$

What does this operation physically correspond to ($f(t)$ represents a distribution of light) ?

5.3 Fourier transform

5.3.1 Fourier transform calculations

Fourier transform of a function or distribution f is :

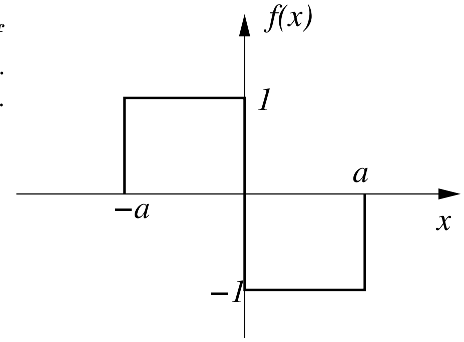
$$\mathcal{F}[f(t)](\nu)=\hat{f}(\nu)=\int_{-\infty}^{\infty} f(t)e^{-2i\pi\nu t}dt$$

We recall some important properties (that you must be able to prove) :

— $\mathcal{F}[f(t+\tau)]=\hat{f}(\nu)e^{2i\pi\nu\tau}$
— $\mathcal{F}\left[e^{2i\pi\nu_0 t}f(t)\right]=\hat{f}(\nu-\nu_0)$
— $\mathcal{F}\left[f(\lambda t)\right]=\frac{1}{|\lambda|}\hat{f}\left(\frac{\nu}{\lambda}\right)$ (for λ real).
— $\mathcal{F}[\overline{f(t)}]=\overline{\hat{f}(-\nu)}$
— $\mathcal{F}\left[\frac{df}{dt}\right]=2i\pi\nu\hat{f}(\nu)$
— $\mathcal{F}[(f*g)(t)]=\hat{f}(\nu).\hat{g}(\nu)$
— $\mathcal{F}[f(t).g(t)]=(\hat{f}*\hat{g})(\nu)$

1. Show that :
   — the FT of a real and even function is real and even
   — the FT of a real and odd function is imaginary and odd
   — $\mathcal{F}[\mathcal{F}[f]](t)=f(-t)$
2. Calculate the FT or FT^{-1} of the following functions (or distributions).
   — $\mathcal{F}[e^{-|t|}]$
   — $\mathcal{F}^{-1}[\Lambda(\nu)]$
   — $\mathcal{F}^{-1}\left[e^{-\pi\nu^2}\right]$
   — $\mathcal{F}[1]$
   — $\mathcal{F}\left[\frac{1}{T}\,Ш(t/T)\right]$
   — $\mathcal{F}^{-1}[\sin 2\pi\nu]$
   — $\mathcal{F}[\cos t]$
   — $\mathcal{F}\left[\frac{\sin^2\pi t}{t}\right]$
   — $\mathcal{F}^{-1}\left[\text{sign}(\nu)\right]$

3. if $g_{\sigma^2}(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$, show that $g_{\sigma^2} * g_{\sigma'^2} = g_{\sigma^2+\sigma'^2}$
4. Calculate the following integrals (from $-\infty$ to $+\infty$) :
$\int \frac{\sin(x)}{x}\,dx$, $\int \frac{\sin^2(x)}{x^2}\,dx$, $\int \frac{\sin(x)\cos(x)}{x}\,dx$, $\int \frac{\sin^2(x)\cos^2(x)}{x^2}\,dx$
5. We consider the signal $f(x)$ described by the graph opposite : f equals 1 between $-a$ and 0, -1 between 0 and a and 0 elsewhere. Calculate its FT $\hat{f}(u)$ and draw its graph (real, imaginary parts).



5.3.2 Operations on signals

1. We consider the following functions : $f_0(t) = \Pi(t/T_0)$; $f_1(t) = f_0(t)\,\sin(\frac{2\pi t}{T})$; $f_2(t) = f_1(t)\sum_n \delta(t - nT_e)$

   with $T_0 = 4T = 12T_e = 24\tau$ $f_3(t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} f_2(t)dt$

   (a) Plot their graph

   (b) Graph the functions $|\hat{f}_0(\nu)|$, $|\hat{f}_1(\nu)|$, $|\hat{f}_2(\nu)|$, $|\hat{f}_3(\nu)|$.

   (c) Comment the operations going from f_0 to f_1 then to f_2 then to f_3 as well as their effects on their FTs.
2. Draw the following functions, for $T_0 = 4T = 16\tau > 0$:
   — $f_0(t) = \Lambda(t/T_0)$
   — $f_1(t) = f_0(t)\,.\sum_{n=-\infty}^{\infty} \delta(t - nT)$,
   — $f_2(t) = f_1(t) * \frac{1}{\tau}\prod\left(\frac{t}{\tau}\right)$
   — Calculate and represent graphs of functions $\hat{f}_0$, $\hat{f}_1$, $\hat{f}_2$, Fourier transforms of functions f_0, f_1, f_2.

5.3.3 Phase contract

Consider the function

$$f(x) = \exp(i\epsilon\cos(2\pi m x))$$

with $\epsilon \ll 1$ and m a positive real number.

1. Calculate $|f(x)|^2$
2. Make a Taylor expansion of f to first order, then write the FT $\hat{f}(u)$. Plot the shape of $\hat{f}$.
3. We multiply $\hat{f}(u)$ by the function $i\Pi(\frac{u}{b})$ with $b \ll m$. The result is noted $\hat{f}_1(u)$. Calculate $\hat{f}_1(u)$.
4. Deduce $f_1(x)$ by inverse FT
5. Calculate $|f_1(x)|^2$ and show that it reproduces the phase variations of f. Discuss in a few sentences the name "phase contrast" given to this technique.

5.3.4 Deconvolution

We consider a signal composed of two Gaussians centered at $\pm b/2$:

$$f(x) = \exp\left(-\pi\frac{(x-b/2)^2}{a^2}\right) + \exp\left(-\pi\frac{(x+b/2)^2}{a^2}\right)$$

with $b > a$.

1. Write $f(x)$ in the form of a convolution between a Gaussian $g(x)$ and a function $h(x)$ which we will explicit
2. Calculate and plot the FT $\hat{f}(u)$ of f
3. We divide $\hat{f}(u)$ by the function $\phi(u) = \exp(-\pi u^2 a^2)$, we call $\hat{f}_1(u)$ the result . Write $\hat{f}_1(u)$ and its inverse FT $f_1(x)$
4. Link $f_1(x)$ to $h(x)$. Why do you think this operation is called a "deconvolution" (explain in a few sentences) ?
5. We now replace the Gaussians by sinc functions, and we consider the following signal :

$$f(x) = \mathrm{sinc}\left(\pi \frac{x - b/2}{a}\right) + \mathrm{sinc}\left(\pi \frac{x + b/2}{a}\right)$$

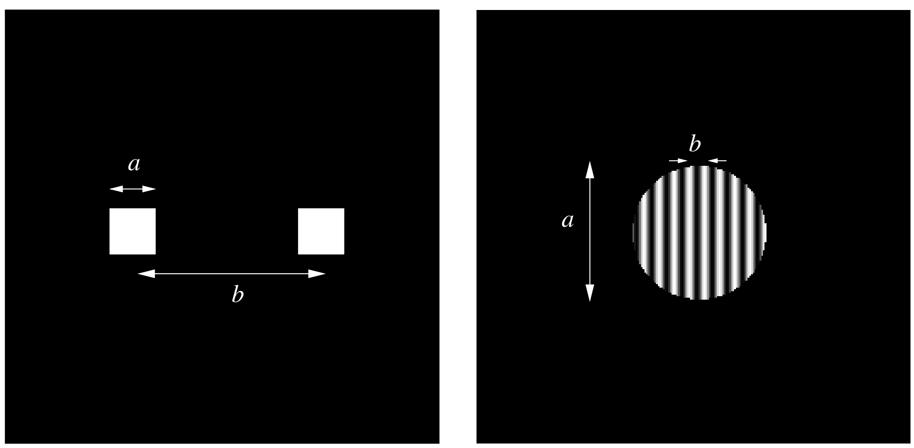
   Repeat questions 1 to 3 (note : in this case the function $\phi(u)$ is no longer a Gaussian but a function to be determined) and explain why the deconvolution does not work in this case.

5.3.5 Optical FT

When a monochromatic light wave with wavelength λ encounters a screen with amplitude transmission coefficient $t(x, y)$, a diffraction phenomenon occurs. At a large distance d the amplitude $f(x, y)$ observed at a point of coordinates (x, y) of the plane $z = d$ are written

$$f(x, y) = K\, \hat{t}\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$$

with K a constant (assumed equal to 1). We will consider two different screens : the first consists of two square slits with side a spaced by b in the direction Ox, the second is a circular diaphragm with diameter a striped with fringes of type $\cos^2$ of period $b \ll a$.



For each screen, write the transmission coefficient $t(x, y)$, calculate the diffracted amplitude $f(x, y)$ and plot the graph of the intensity $|f(x, y)|^2$ (take $b = 10a$ for the first mask and $a = 10b$ for the second).

5.3.6 Recording a sound

We consider a singer on the stage of an opera. The singer produces a sound signal of intensity $f(t)$, t being the time. We assume that the singer is able to produce sounds of frequencies ν such that $|\nu| \in [\nu_1, \nu_2]$.

1. Drawing inspiration from the data in the statement, roughly draw the appearance of the real and imaginary parts of the Fourier transform (TF) $\hat{f}(\nu)$ (ν can be positive or negative). Put on the graphs as many indications as possible.
2. A microphone records the singer's voice. The microphone is characterized by a causal impulse response that we will model by the function $h(t) = \frac{H(t)}{\tau} \exp(-t/\tau)$ ($\tau > 0$). Calculate $\hat{h}(\nu)$ and draw schematically its real and imaginary parts.
3. We call $g(t)$ the recorded sound signal. What is the relationship between the original signal f and the recorded signal g ?
4. What becomes of this relation in Fourier space ? Graph the shape of the real parts of $\hat{f}$, $\hat{g}$, $\hat{h}$ putting in evidence the frequency filtering introduced by the microphone.

5. In your opinion (justify your opinion with scientific arguments) is the microphone a high-pass filter ? low pass ? band pass ? of another nature (specify) ? What is the condition on τ so that the highest frequency of f is transmitted by the microphone (we will consider that a frequency ν is transmitted if the ratio $|\hat{g}(\nu)/\hat{f}(\nu)| > 0.001$

5.3.7 Linear filters

1. Consider a filter whose impulse response is :

$$R(t) = \sin^2(\pi t) \prod \left(t - \frac{1}{2} \right)$$

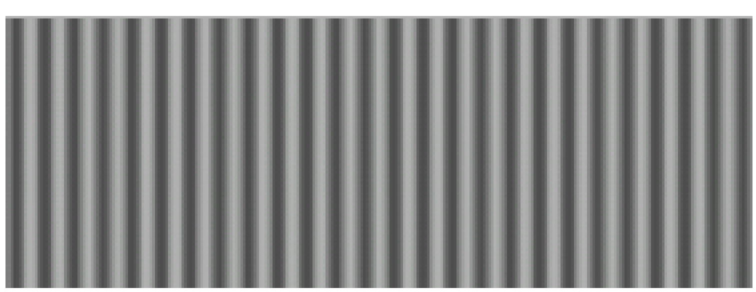
   (a) Is it a causal filter ? Make a plot of $R(t)$.
   (b) Let $x(t)$ be an input signal. Write the output signal $y(t)$ as a time integral.
   (c) Express the output $y(t)$ as a frequency integral. Calculate $\hat{R}(\nu)$ (hint : use $\sin^2 x = (1 - \cos 2x)/2$). Schematically represent the graph of $\hat{R}(\nu)$.
   (d) Is it a low pass or high pass filter ? How does this filter compare with one whose impulse response is $R_0(t) = \Pi(t - \frac{1}{2})$?

2. We consider a linear filter acting on a time signal $x(t)$. This filter is characterized by a transfer function $f(\nu)$ (ν is the frequency). The filtered signal is called $y(t)$.
   (a) Recall the so called “impulse response” of the filter. Does it depend on the signal x ? signal y ? of the function f ? What is the impulse response in our case (to be written with only the data of the statement) ?
   (b) Write the relation between the signals $x(t)$ and $y(t)$ (direct plane). How is this relation written in the Fourier plane ?
   (c) The transfer function is of the form $f(\nu) = h_0 \exp(-2i\pi\tau\nu)$ with h_0 and τ positive reals. What is the relation in the direct plane between x and y in this case ? Give a physical example of this type of filtering. Does the case $\tau < 0$ have a physical significance and why ?
   (d) Same question if the transfer function is $K\nu$, with K a pure imaginary constant.
   (e) Same question if the transfer function is $h_0\delta(\nu - \nu_0)$ (h_0 and ν_0 positive reals). Show that in this case the filtered signal y always has the same form. What happens if $\nu_0 = 0$?
   (f) We now assume a transfer function of the form $h_0 \prod \left(\frac{\nu}{\nu_0} \right)$ (ν_0 positive real).
      i. What is the associated impulse response $h(t)$? Draw its graph.
      ii. What is the limit of $h(t)$ when $\nu_0 \to 0$? What then is the relation between x and y in the direct plane ?
      iii. Same questions if $\nu_0 \to \infty$.
      iv. For any ν_0, write the signal $y(t)$ when $x(t) = x_0\, \delta(t - \tau)$ (τ real).
      v. Recall the relation between $\delta(t)$ and the Heaviside unit step $H(t)$, deduce an integral form for $y(t)$ in the case where $x(t) = x_0\, H(t - \tau)$. Inspired by the graph of $h(t)$, roughly draw that of $y(t)$.

5.3.8 Optical filtering

We consider a sinusoidal pattern described by the function

$$O(x, y) = 1 + m \cos \left(\frac{2\pi x}{a} \right)$$



with $m \ll 1$. We carry out an optical filtering experiment whose transfer function is written

$$T(u,v) = \exp\left(i\pi K(u^2+v^2)\right)$$

with K a constant. The amplitude of the filtered object is denoted $f(x,y)$ and is calculated thanks to the filtering relation in the Fourier plane : $\hat{f}(u,v) = \hat{O}(u,v)\,T(u,v)$. Calculate this amplitude and the corresponding intensity $|f(x,y)|^2$. Show that there exist values of K for which the intensity is uniform, and that we may have contrast inversions (inverted black and white stripes).

5.4 Correlation and power spectrum

1. Calculate the power spectrum and the autocorrelation of $f(t) = a\cos(2\pi\nu_0(t-t_0))$, show that it is independent of origin t_0. How these functions are modified if f is multiplied by a Gaussian $g(t) = \exp(-\pi t^2/a^2)$?
2. Calculate the autocorrelation of $f(x) = \delta(x) + m\delta(x-a)$ then that of $g(x) = \Pi(x) + m\Pi(x-a)$. What is the relationship between $C_f(\rho)$ and $C_g(\rho)$?
3. Let the signal $f(t) = a_1\cos(2\pi\nu_1 t) + a_2\cos(2\pi\nu_2 t)$ with ν_1 and ν_2 very close and $\nu_2 > \nu_1$. We set $\delta\nu = \nu_2 - \nu_1$ and $\nu_0 = (\nu_1+\nu_2)/2$.
   — Calculate the power spectrum $W_f(\nu)$ of f
   — Same question if the signal is truncated by a gate of width $T \gg 1/\nu_1$.
   — What must be the minimum value of T if we want to "resolve" the two frequencies ν_1 and ν_2 ? N.A. : we observe the oscillations of the Sun ($f(t)$ is the corresponding signal) during a continuous period of time of 12h. What is, in Hz, the corresponding resolution on the power spectrum of f ?
4. Calculate the autocorrelation of a chirp function $f(t) = \cos(\pi x^2/a^2)$
5. We consider the sum
$$f(x) = \sum_{n=0}^{N} a_n\,\delta(x-x_n)$$
with x_n random numbers uniform distributed between 0 and 1. Show that the autocorrelation of f tends to a Dirac when $N \to \infty$. What happens to this autocorrelation if we replace $\delta(x-x_n)$ by $g(x-x_n)$ with g any function (relate $C_f(\rho)$ and $C_g(\rho)$) ?
6. We consider two signals $f(t)$ and $g(t) = f(t-a)$. Explain how the phase of the cross-spectrum of f and g makes it possible to measure the shift a between f and g. Relate $C_{fg}(\tau)$ and $C_f(\tau)$.
7. A centered white noise is a function whose each value $f(t)$ is a random number with zero mean, and whose power spectral density is the constant σ^2 (variance of the random numbers that make up f). Consider the sum $g(t) = f(t) + a\cos(\omega_0 t)$. Calculate the autocorrelation of g and show that it can detect the sinusoid even if it is drowned in noise ($a < \sigma$).
8. Using Parseval's theorem, calculate the following integrals :
$\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ and $\displaystyle\int_{-\infty}^{\infty} \mathrm{sinc}(\pi x)^n\,dx$ for $n = 2,3,4$
9. We consider a *slowly* damped vibration having the expression :
$$V(t) = \begin{cases} 0 \quad \text{if } t<0 \\ A\exp\left(-\frac{t}{\tau}\right)\cos(2\pi\nu_0 t) \quad \text{if } t>0 \end{cases}$$
τ being the relaxation time of the vibration, which is assumed to be long compared to the signal period ($\nu_0 \gg 1/\tau$). What is the (complex) frequency spectrum $\hat{V}(\nu)$ of this vibration. Deduce the power spectrum and find the relation between τ and the width at half maximum $\delta\nu$ of the peaks of $|\hat{V}(\nu)|^2$.
10. Determine the functions $\varphi_n(t)$ such as :
$$\hat{\varphi}_n(\nu) = \Pi\left(\frac{\nu}{2B}\right)\exp\left(-i\pi n\frac{\nu}{B}\right).$$
Using the Plancherel-Parseval formula, calculate
$$\int_{-\infty}^{\infty} \overline{\varphi}_n(t)\varphi_m(t)\,dt$$
What can we deduce ?

5.5 Fourier series

Any periodic signal $f(t)$ of period T can be expressed as a sum of sines and cosines, or equivalently as a sum of complex exponentials. The expansion is of the form :

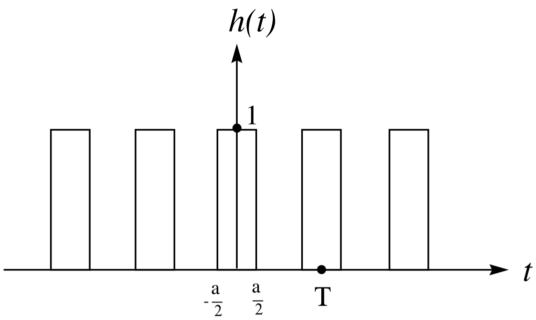
$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{i2\pi n \frac{t}{T}}$$

the coefficients c_n are :

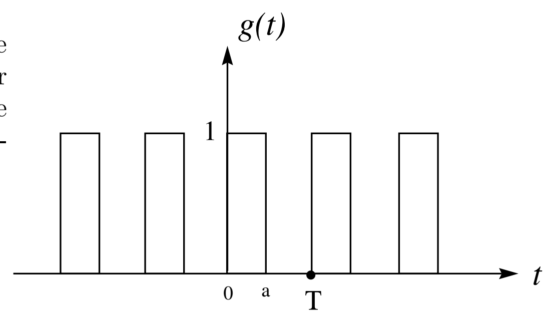
$$c_n = \frac{1}{T}\int_0^T f(t)e^{-i2\pi n \frac{t}{T}}\,dt$$

The set of complex numbers c_n constitutes what is called the *frequency spectrum* of the signal $f(t)$.

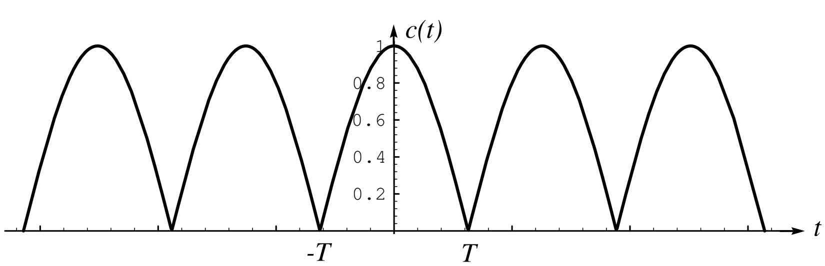
1. Show that we can define the c_n in an equivalent way by calculating the above integral with integration bounds between t_0 and $t_0 + T$ (in particular between $-T/2$ and $T/2$).
2. Show that if $f(t)$ is even (and real), the c_n are purely real numbers. Subsequently the Fourier series can be written as a sum of cosines only. What happens if $f(t)$ is odd ?
3. Calculate and graph the frequency spectrum of the crenel signal $h(t)$ drawn below.



4. Calculate and graph the frequency spectrum of the crenel signal $g(t)$ drawn below. (Show the factor $\exp(-i\pi na/T)$ in the Fourier coefficients). Compare the spectra of $h(t)$ and $g(t)$. Deduce a general underlying property.



5. A sawtooth signal is obtained by periodizing the linear function $f(t) = at$, $(0 \leq t < T)$. Calculate the frequency spectrum of this signal
   — by direct method,
   — using the property that $f'(t)$ is a Dirac comb.
   Compare this development with that of another sawtooth signal, defined by $\sum_n \Lambda(\frac{t-2n}{T})$.
6. Give the Fourier series expansion of the following function (cosine absolute value) :



7. Relate the Fourier series of a real signal, expressed in terms of complex exponentials, to the Fourier series written using sine functions and cosine where all coefficients are real.
8. We consider the function

$$\hat{f}(\nu) = \frac{\hat{g}(\nu)}{(2\pi i\nu)^2},$$

with $\hat{g}(\nu) = 1 - 3\mathrm{e}^{-i2\pi\nu} + \frac{5}{2}\mathrm{e}^{-i4\pi\nu} - \frac{1}{2}\mathrm{e}^{-i8\pi\nu}$.

— Check by calculation that the limit $\nu = 0$ exists for $\hat{f}(\nu)$.
— Calculate and plot $g(t)$.
— Deduce $f(t)$ and represent its graph. Check that $\hat{f}(0)$ is the value we expect.

9. Compute the mean and the autocorrelation $\Gamma_f(\tau)$ of the following functions :

   $f(t) = e^{i\omega t}$, $f(t) = \Pi(t)$, $f(t) = \delta(t-a) + \delta(t+a)$, $f(t) = \displaystyle\sum_{n=-\infty}^{\infty} \varphi(t - n\tau)$ with $\varphi(t) = \Pi(t)$ and $\varphi(t) = \delta(t)$ (use the Poisson relation between $f(t)$ and its Fourier series).

   We recall that $\langle f \rangle = \dfrac{1}{T}\displaystyle\int_0^T f(t)\,dt$ and that $\Gamma_f(\tau) = \dfrac{1}{T}\displaystyle\int_0^T \bar{f}(t) f(t+\tau)\,dt$ for a periodic function of period T. For nonperiodic functions, we will simply calculate the integrals from $-\infty$ to ∞ without multiplying by $1/T$.

5.6 Sampling and periodization

1. Let $f(t)$ be any signal which decays very rapidly outside the interval $[0, T]$. Represent the graphs of the following functions :
   — $f_{\text{per}} = f(t) * Ш_T(t)$.
   — $Ш_T(t) * \Pi(t/\tau)$, with $T = 3\tau$.
   — $f_{\text{ech}} = f(t)\, Ш_T(t)$.
   — $g(t) = [f(t) * Ш_T(t)]\, Ш_{T_e}(t)$, with $T = NT_e$.
   — $f_{\text{ech}} * \Pi(\frac{t}{T})$.
2. From the sampled values of a function $f(t)$, one can construct a continuous function $f_a(t)$ which approximates $f(t)$ piecewise linearly. Show that the following expression performs this *linear interpolation* operation of f :

$$f_a(t) = \Lambda(t/T) * f_{\text{ech}}(t)$$

5.6.1 Convergence acceleration of a series

We recall the relation $\displaystyle\sum_{-\infty}^{\infty} f(n) = \sum_{-\infty}^{\infty} \hat{f}(n)$ (deduced from the Poisson's formula).

1. Consider the series

$$S = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

   Write the main term of the series using a cardinal sine function f, then calculate S using the Fourier transform $\hat{f}$.
2. Calculate the sum $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$

5.6.2 Sampling a narrowband signal

The signal $x(t)$ is said to be *narrow band* if its FT $\hat{x}(\nu)$ is zero outside the intervals $|\nu - \nu_0| < \beta$ and $|\nu + \nu_0| < \beta$, with condition $\beta \ll \nu_0$. The questions below aim to analyze the problem of sampling this type of signal. We assume $x(t)$ real.

1. Give the qualitative appearance of the function $Re\ \hat{x}(\nu)$. Show that the direct application of Shannon's theorem would lead to choosing a sampling frequency $\nu_e = 2(\nu_0 + \beta)$.
2. We consider the *analytical signal* associated with $x(t)$:

$$\begin{aligned} z(t) &= 2\int_0^{\infty} \hat{x}(\nu) e^{2\pi i \nu t}\, d\nu \\ &= x(t) + iy(t) \end{aligned}$$

   Represent $Re\ \hat{z}(\nu)$ on the graph of the previous question. Show that $z(t)$ can be expressed as $z(t) = A(t)\exp(2\pi i \nu_0 t)$, where $A(t)$ is a low signal frequency. Represent $\hat{A}(\nu)$.
3. Let $A(t) = A_1(t) + iA_2(t)$. Using the relation $x(t) = Re\ z(t)$, express $x(t)$ in terms of A_1 and A_2.

4. Apply Shannon's sampling formula to the signal $A(t) = z(t)\exp(-2\pi i\nu_0 t)$. Deduce a formula which makes it possible to reconstruct the signal $x(t)$ from the samples $A_1(t_n)$ and $A_2(t_n)$ at instants $t_n = \frac{n}{2\beta}$. What can we conclude with regard to question 1 ?
5. Show that in practice, $A(t)$ can be obtained in the following way : $A_1(t)$ is obtained by low-pass filtering of the signal $u(t) = 2\cos(2\pi\nu_0 t)x(t)$. Similarly $A_2(t)$ is obtained from a low-pass filtering of $v(t) = 2\sin(2\pi\nu_0 t)x(t)$.

5.6.3 Link between the Shannon formula and the Fourier series

1. Consider a signal $v(t)$ whose Fourier transform is defined as follows : $\hat{v}(\nu) = a|\nu|$ if $(-\nu_0 < \nu < \nu_0)$ and $\hat{v}(\nu) = 0$ otherwise.
   - How should we sample $v(t)$ to be able to reconstruct this signal without loss of information ?
   - Calculate $v(t)$ and write Shannon's formula explicitly in the particular case of this signal.
   - Compute the Fourier transform of the two sides of the resulting equation. What do we find ?
2. Generalization of Shannon's formula.

   We consider the trapezium function defined as follows :

$$\hat{M}(\nu) = \frac{B+\beta}{\beta}\Lambda\left(\frac{\nu}{B+\beta}\right) - \frac{B}{\beta}\Lambda\left(\frac{\nu}{B}\right)$$

   where B and β are two positive numbers.
   - Plot the graph of $\hat{M}(\nu)$, with $\beta = B/2$.
   - Calculate $M(t)$.
     hint : use the formula $\sin^2 a - \sin^2 b = \sin(a+b)\sin(a-b)$.
   - Let $f(t)$ be a band-limited signal $[-B, B]$. Explain, using an appropriate graph, why we can write :

$$\hat{f}(\nu) = \left[\hat{f}(\nu) * \frac{1}{2B+\beta}\,\text{Ш}\left(\frac{\nu}{2B+\beta}\right)\right].\hat{M}(\nu)$$

   - Deduce the following formula :

$$f(t) = \sum_n f\left(\frac{n}{2B+\beta}\right)\,\text{sinc}(\pi([2B+\beta]t - n))\,\text{sinc}\left(\pi\left(\beta t - \frac{\beta n}{2B+\beta}\right)\right)$$

     How does this expression compare to the "classic" Shannon formula ? How can this generalization be advantageous ?
3. Let $f(t)$ be a real band-limited signal : $\hat{f}(\nu) = 0$ for $|\nu| \geq B$. Let $E(t)$ be the periodic signal defined by $E(t) = \sum_{n)-\infty}^{\infty} A\prod\left(\frac{t-nT}{\tau}\right)$, where $T \geq \tau$. We set :

$$f_E(t) = f(t)E(t)$$

   - Draw an example of the graphs of $f(t)$, $E(t)$ and $f_E(t)$.
   - Connect the Fourier transforms $\hat{f}_E(\nu)$ and $\hat{f}(\nu)$.
   - Draw the graph of $\hat{f}_E(\nu)$ in the case $\tau/T = 1/2$, $T = \frac{1}{2B}$, $\hat{f}(\nu) = \Lambda(\nu/B)$.
   - Explain how it is possible, using a suitable filter, and under the Shannon condition, to reconstruct the function $f(t)$ from the function $f_E(t)$. Call $R(t)$ the impulse response of the filter such that $f(t) = R(t) * f_E(t)$.
   - Using the last equation, in what limit do we recover Shannon's sampling theorem ?
4. Let $f(t)$ be a narrowband signal with respect to ν_0, i.e. $\hat{f}(\nu) \neq 0$ only for $|\nu| \ll \nu_0$. We consider the recurrence defined by the following operations on the signal $f(t)$:

$$\begin{aligned} f_1(t) &= 2\cos(2\pi\nu_0 t)f(t) \\ f_2(t) &= 2\cos(4\pi\nu_0 t)f_1(t) \\ f_3(t) &= 2\cos(8\pi\nu_0 t)f_2(t) \end{aligned}$$

   (a) Compute $\hat{f}_3(\nu)$ as a function of $\hat{f}(\nu)$ and represent the result schematically on a graph.

(b) What is $\lim_{m\to\infty} \hat{f}_m(\nu)$ if we continue the sequence of multiplication by cosines defined by $f_m(t) = 2\cos(2^m \pi \nu_0 t) f_{m-1}(t)$?

(c) We consider the periodization

$$\hat{f}_{\text{per}}(\nu) = \sum_n \hat{f}\left(\nu - \frac{(2n+1)}{T_0}\right)$$

where $T_0 = 1/\nu_0$. Calculate the inverse Fourier transform by displaying a Dirac comb and plot the (real) distribution obtained. We recall that $f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$.

Appendix :
Fourier transforms

Definitions

Direct FT

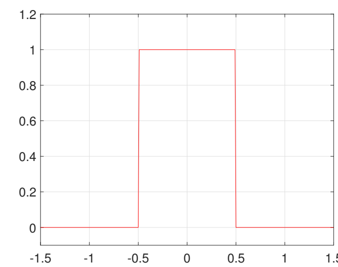
$$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-2i\pi\nu t}\, dt$$

Inverse FT

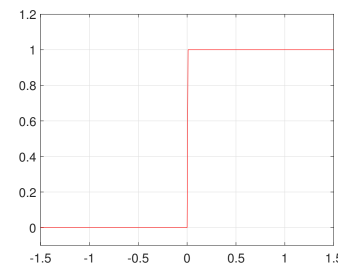
$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\nu)\, e^{2i\pi\nu t}\, d\nu$$

Miscellaneous

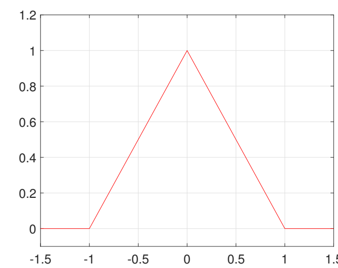
— Gate (or rectangular) function : $\Pi(t) = \begin{cases} 1 \text{ if } |t| < 1/2 \\ 0 \text{ if } |t| > 1/2 \end{cases}$



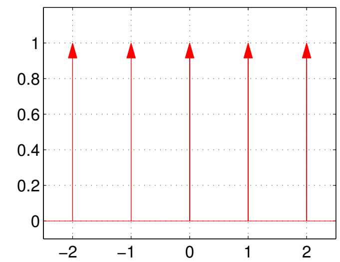
— Heaviside function : $H(t) = \begin{cases} 1 \text{ if } t > 0 \\ 0 \text{ if } t < 0 \end{cases}$



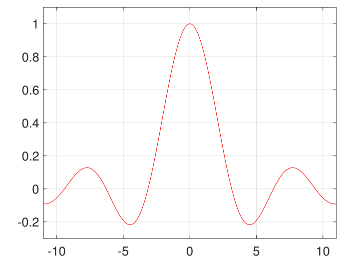
— triangle function : $\Lambda(t) = \begin{cases} 1 - |t| \text{ if } |t| \leq 1 \\ 0 \text{ otherwise} \end{cases}$



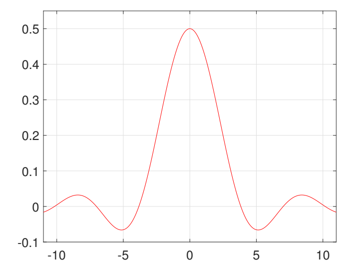
— Dirac comb : $Ш(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$



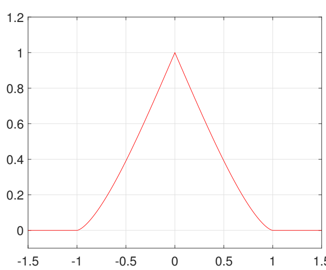
— Cardinal sine : $\mathrm{sinc}(t) = \dfrac{\sin t}{t}$



— Cardinal J_1 : $\mathrm{jinc}(t) = \dfrac{J_1(t)}{t}$



— Chinese hat : 大$(t) = \dfrac{2}{\pi}\left[\mathrm{arcos}(|t|) - |t|\sqrt{1-t^2}\right]$



General properties

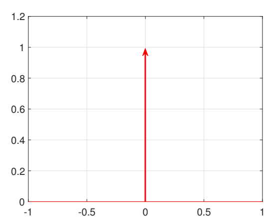
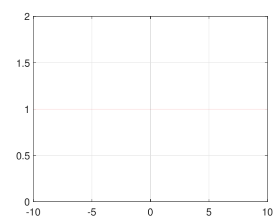
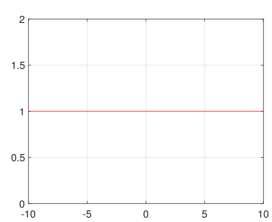
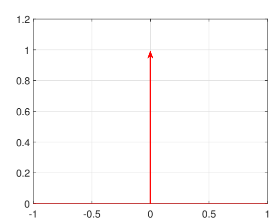
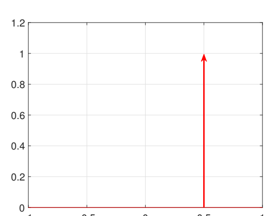
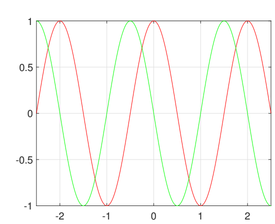
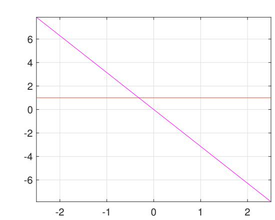
Function	Transform	Function	Transform
$a\,f(t) + b\,g(t)$	$a\,\hat{f}(\nu) + b\,\hat{f}(\nu)$	$\dfrac{df}{dt}$	$2i\pi\nu\hat{f}(\nu)$
$f\left(\dfrac{t}{a}\right)$	$\|a\|\,\hat{f}(a\nu)$	$t.f(t)$	$-\dfrac{1}{2i\pi}\,\dfrac{d\hat{f}}{d\nu}$
$\bar{f}(t)$	$\overline{\hat{f}(-\nu)}$	$f(t).g(t)$	$(\hat{f} * \hat{g})(\nu)$
$f(t+\tau)$	$\hat{f}(\nu)\,e^{2i\pi\nu\tau}$	$(f * g)(t)$	$\hat{f}(\nu).\hat{g}(\nu)$
$e^{2i\pi\nu_0 t} f(t)$	$\hat{f}(\nu - \nu_0)$	$C_{fg}(t) = \int \overline{f(\tau)}\, g(t+\tau)d\tau$	$\hat{h}(\nu) = \overline{\hat{f}(\nu)}\,\hat{g}(\nu)$

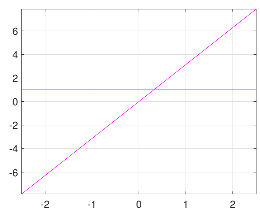
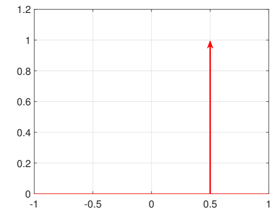
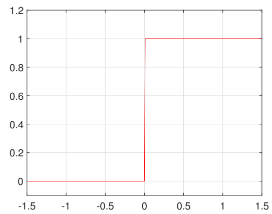
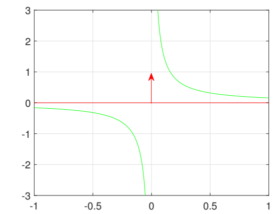
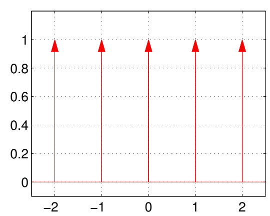
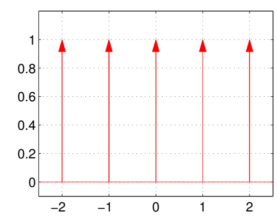
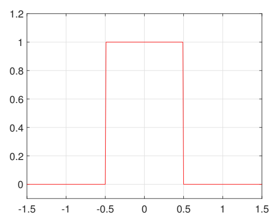
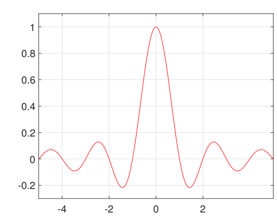
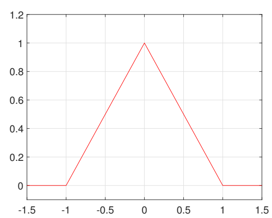
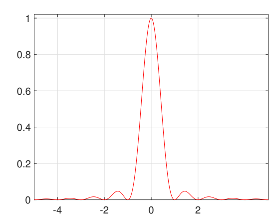
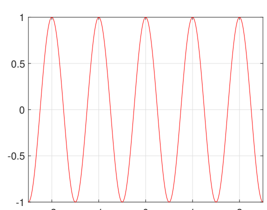
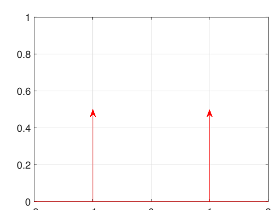
- $\hat{f}(0) = \int_{-\infty}^{\infty} f(t)dt$ $\qquad f(0) = \int_{-\infty}^{\infty} \hat{f}(\nu)d\nu$ $\qquad \hat{\hat{f}}(t) = f(-t)$
- Parseval's theorem $\qquad \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \int_{-\infty}^{\infty} \hat{f}(\nu)\,\overline{\hat{g}(\nu)}\,d\nu$
- Périodic functions : Poisson summation formula (Fourier series) $\qquad \sum_{n=-\infty}^{\infty} \phi(x-na) \;=\; \sum_{n=-\infty}^{\infty} \hat{\phi}\left(\frac{n}{a}\right)\, e^{\frac{2i\pi nx}{a}}$

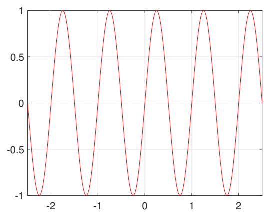
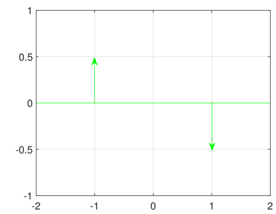
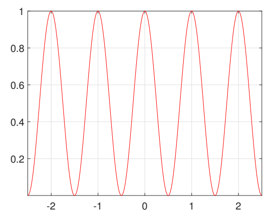
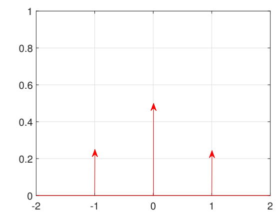
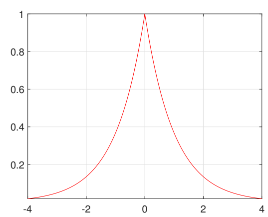
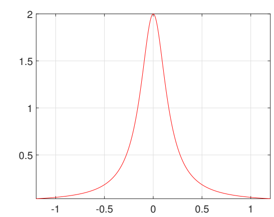
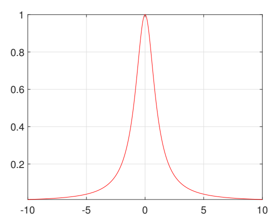
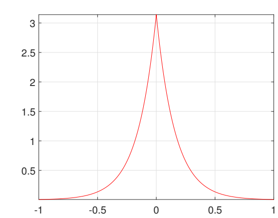
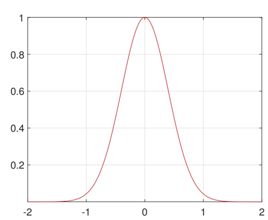
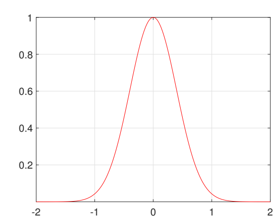
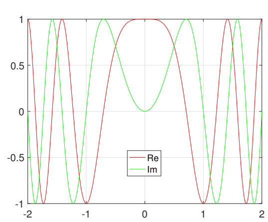
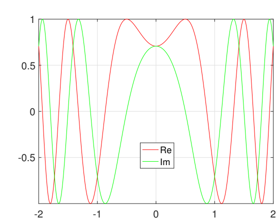
Parity related properties (real functions only)

f real and even	$\Longleftrightarrow$	$\hat{f}$ real and even
f real and even	$\Longleftrightarrow$	Fourier transform equals inverse Fourier transform
f real and odd	$\Longleftrightarrow$	$\hat{f}$ pure imaginary and odd
f any real function	$\Longleftrightarrow$	$\hat{f}(-u) = \overline{\hat{f}(u)}$ (even real part, odd imaginary part)

Functions of one variable

Function	Graph	Transform	Graph
$\delta(t)$		1	
1		$\delta(\nu)$	
$\delta(t-\tau)$		$\exp(-2i\pi\nu\tau)$ – real part – imaginary part – modulus – phase	 

Function	Graph	Transform	Graph
$\exp(2i\pi mt)$		$\delta(\nu - m)$	
$H(t)$		$\frac{1}{2}\delta(\nu) + \mathrm{VP}(\frac{1}{2i\pi\nu})$	
$Ш(t)$		$Ш(\nu)$	
$\Pi(t)$		$\mathrm{sinc}(\pi\nu)$	
$\Lambda(t)$		$\mathrm{sinc}^2(\pi\nu)$	
$\cos(2\pi mt)$		$\frac{1}{2}\delta(\nu - m) + \frac{1}{2}\delta(\nu + m)$	

Function	Graph	Transform	Graph
$\sin(2\pi mt)$		$-\frac{i}{2}\delta(\nu-m)+\frac{i}{2}\delta(\nu+m)$	
$\cos^2(\pi mt)$		$\frac{1}{2}\delta(\nu)+\frac{1}{4}\delta(\nu-m)+\frac{1}{4}\delta(\nu+m)$	
$\exp(-\lvert t\rvert)$		$\frac{2}{1+4\pi^2\nu^2}$	
$\frac{1}{1+t^2}$		$\pi\ \exp(-2\pi\lvert\nu\rvert)$	
$\exp(-\pi t^2)$		$\exp(-\pi\nu^2)$	
$\exp\left(i\pi\frac{t^2}{a^2}\right)$		$\sqrt{i}\lvert a\rvert\ \exp(-i\pi a^2\nu^2)$	

Functions of two variables

Direct FT

$$\hat{f}(u,v) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-2i\pi(ux+vy)}\, dxdy$$

Inverse FT

$$f(x,y) = \iint_{-\infty}^{\infty} \hat{f}(u,v)\, e^{2i\pi(ux+vy)}\, dudv$$

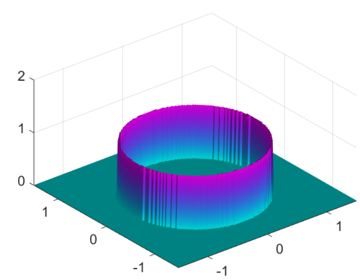
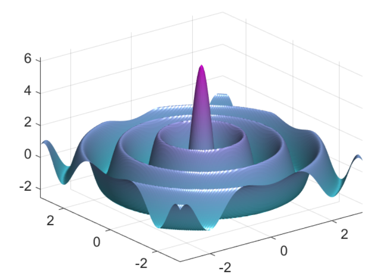
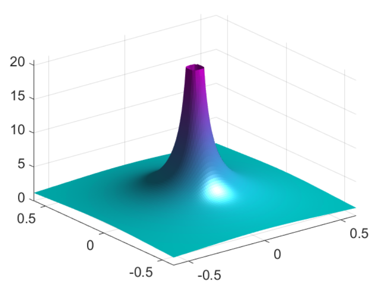
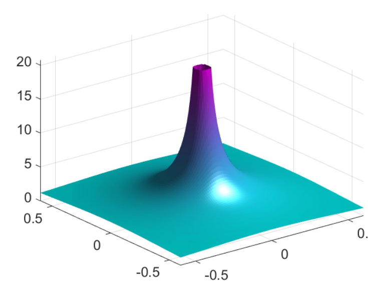
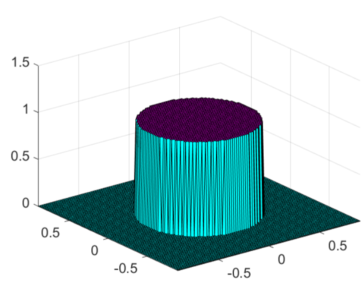
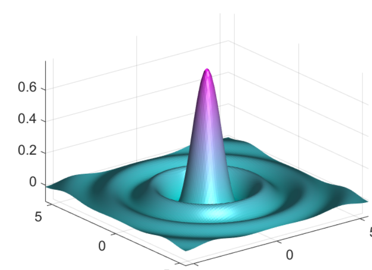
Separable functions $f(x,y) = g(x)\,h(y) \Longleftrightarrow \hat{f}(u,v) = \hat{g}(u)\,\hat{h}(v)$

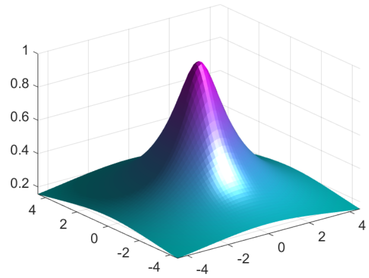
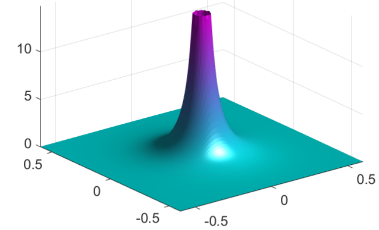
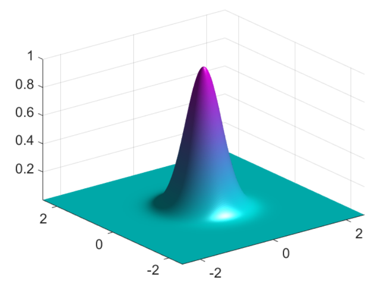
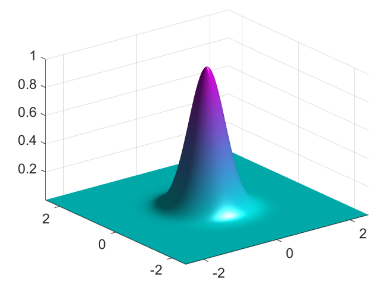
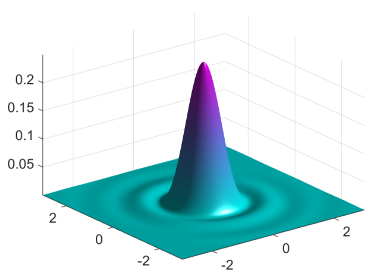
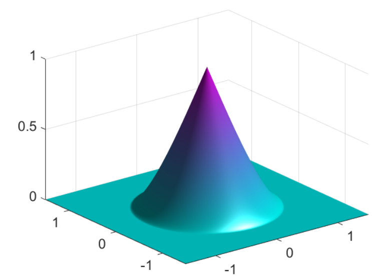
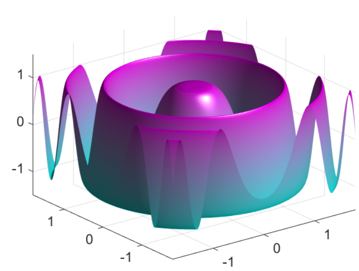
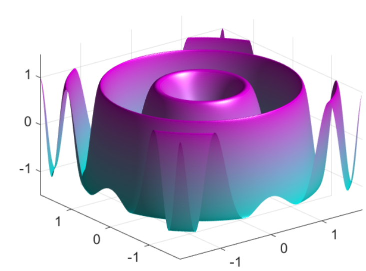
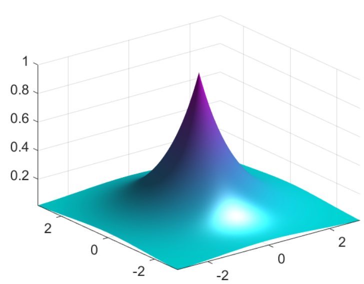
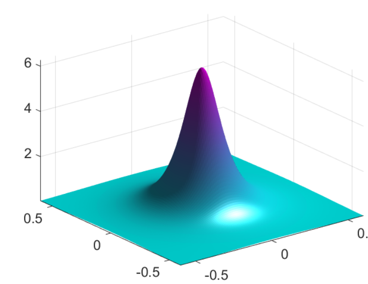
Functions with circular symmetry
Let $r = \sqrt{x^2+y^2}$ and $q = \sqrt{u^2+v^2}$. The 2D direct and inverse FT have also circular symmetry, and can be expressed as Hankel transforms :

$$\hat{f}(q) = \int_0^{\infty} f(r)\, J_0(2\pi qr)\, 2\pi r\, dr$$

and

$$f(r) = \int_0^{\infty} \hat{f}(q)\, J_0(2\pi qr)\, 2\pi q\, dq$$

Function	Graph	Transform	Graph
$\delta(r-a)$		$2\pi a J_0(2\pi aq)$	
$1/r$		$1/q$	
$\Pi\left(\frac{r}{d}\right)$		$2\left(\frac{\pi d^2}{4}\right)\,\mathrm{jinc}(\pi dq)$	

Function	Graph	Transform	Graph
$\dfrac{1}{\sqrt{r^2+a^2}}$		$\dfrac{\exp(-2\pi aq)}{q}$	
$\exp(-\pi r^2)$		$\exp(-\pi q^2)$	
Airy disc $\mathrm{jinc}(\pi ar)^2$ $(a>0)$		$\dfrac{1}{\pi a^2}$大$\left(\dfrac{q}{a}\right)$	
$\exp\left(i\pi\dfrac{r^2}{a^2}\right)$	 *(real part)*	$ia^2\exp(-i\pi a^2q^2)$	 *(real part)*
$\exp(-ar)$		$\dfrac{2\pi a}{(4\pi^2q^2+a^2)^{3/2}}$	
$r^2\,f(r)$		$\nabla^2\,\hat{f}$	